Spectral Properties of Weakly Inhomogeneous BCS-Models in Different Representations

Michael Benner and Alfred Rieckers

Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany

Reprint requests to Prof. A. R.; E-mail: alfred.rieckers@uni-tuebingen.de

Z. Naturforsch. **60a,** 343 – 365 (2005); received February 16, 2005

For a class of Bardeen-Cooper-Schrieffer (BCS)-models, with complex, weakly momentum dependent interaction coefficients, the representation dependent effective Hamiltonians and their spectra are reconsidered in order to obtain a consistent physical picture by means of operator algebraic methods. The starting point is the limiting dynamics, the existence of which had been proved in a previous work, in terms of a C*–dynamical system acting in a classically extended, electronic Canonical Anticommutation Relations (CAR)-algebra. The C*-algebraic KMS–theory, including the low temperature limit, specifies the order parameters. These appear as classical observables, which commute with all other observables, constituting elements of the center of the algebra. The algebraic spectral theory, in the sense of Arveson, is first applied to the dynamics in general pure energy state representations. The spectra of the finite temperature representations are analyzed, identifying the gap as the lowest of those energy values, which are stable under local perturbations. Further insights are obtained by decomposing the thermal dynamical systems into the pure energy state Heisenberg dynamics, after having first extended them to more comprehensive W*-dynamical systems. The decomposing orthogonal measure is transferred to the infinite product space of quasi-particle occupation numbers and its support is characterized in terms of 0-1-laws leading to an asymptotic ratio of quasi-particles and holes, which depends on the temperature. This ratio is connected with an algebraic invariant of the representation dependent observable algebra. Energy renormalization aspects and pair occupation probabilities are discussed. The latter reveal, beside other things, the difference between macroscopic term occupation and coherent macroscopic term occupation for a condensate.

Key words: Superconductivity; Phase Transitions; Energy Renormalization; Arveson Spectrum.

1. Introduction

For the theoretical treatment of superconductors Bardeen-Cooper-Schrieffer (BCS)-like models are still of actual interest. Even for the high-T_c materials, modified BCS-interactions are considered or serve at least as reference models. The condensed state of Cooper pairs is a paradigm for many condensation phenomena in quantum field theory. Thus it seems worth while to invest some efforts to elucidate, as much as possible, the structure and typical properties of those models.

For our present discussion we make use of a class of inhomogeneous BCS-models (that is, with momentum dependent interaction terms), which are still explicitly treatable, but which display already the typical spectral features of the quasi-particle and pair excitations. Like all of the BCS-models in theoretical physics (c f., e. g., [1 – 8]) our models are of meanfield type. This feature is here, however, not introduced by replacing certain interaction operators by c-numbers in the middle of a calculation, but by the very definition of the models. That means that we make an ansatz for the microscopic pairing interactions, which coincides for a fixed volume with usual momentum dependent field interactions, but which displays a certain scaling behaviour in the thermodynamical limit. This scaling amounts to an averaging procedure of some of the interaction operators. The formally averaged operators commute with all other observables and enable, by this, the evaluation of the models.

On the other hand, there arise nontrivial mathematical problems by the fact that the averaging procedure converges only in a rather weak topology and not in the algebraic norm topology. The weak topology is commonly obtained by selecting a Hilbert space representation for the observable algebra, mostly a representation over a pure phase temperature state. This connects, on one hand, the dynamics with special external parameters and prevents, on the other hand, the dynamical description of processes far from equilibrium.

In previous publications [9, 10] we have elaborated an alternative method, by which we extend the norm-closed quasilocal CAR-algebra (based on the Canonical Anticommutation Relations for the conducting electrons near the Fermi surface) to a C*-algebra with a non-trivial center. By this we are able to describe the Heisenberg dynamics in the thermodynamic limit in form of a C*-dynamical system, that means a representation independent form of the dynamics. (Mathematically a C*-dynamical system is a one-parameter group of transformations, which act on the observables in a linear, multiplicative and $*$-preserving way and which depend on the time parameter in a strongly continuous manner [11]). The merits of this formulation, the mathematical technicalities of which are given in [9] and [10], are a unified theoretical frame for the various situations resp. reservoir couplings, a superconductor may encounter. Especially, the very powerful C*-algebraic KMS-theory may be applied for all kinds of external parameters, and its low temperature limits can be studied for one and the same dynamical system.

For the characterization of the superconducting state the spectral features of its excitations are essential. In usual many body physics this mostly is discussed in terms of thermal Green's functions using formal perturbation theory. In the spectral representation of Green's functions the poles are, however, sensitive also to small perturbations of the interaction, whereas the temperature dependent energy gap should represent stable features of the collective phenomenon. In a pure Green's function formalism one has also lost the connection to the well developed spectral theory for operators. In cases, where a Hilbert space representation of the many body model is available, there may arise discrepances between the spectrum of the Green's functions and the operator spectrum (personal communication by H. Stumpf).

We aim to clarify such spectral structures by making a clear distinction between the algebraic spectrum of the abstract C*-dynamical system and the spectra of the various unitary implementations in the (inequivalent) representation spaces. Since we have available, in contrast to other meanfield models in algebraic quantum theory, the microscopic, abstract Heisenberg dynamics, we are able to give a deductive treatment of the collective phenomena in the special representation spaces. The identification of the latter is an integral part of a rigorous model discussion in many body physics and depends on both the interactions and the external parameters and reservoirs. This systematic discussion, starting from the universal abstract dynamics, is certainly not the easiest way for arriving at the physical results. But we hope that the present treatment gives an idea how also more difficult questions may be dealt with by means of algebraic quantum field theory.

Since the abstract spectral theory as developed in the seminal paper of Arveson [12] (cf. also [11]) is not well known, we describe its basic definitions and properties – in a necessarily concise form – in Appendix A. By means of the C*-dynamical system τ, e.g., an abstract excitation operator A (creation- or annihilation-operator of the field algebra or rising- or lowering-operator of a Lie algebra) acquires a spectrum $Sp^{\tau}(A)$. In Appendix B we show that $Sp^{\tau}(A)$ comprises all unitary spectra $Sp^{U}(A\Omega)$ of excitations of the vacua Ω, the mathematical definition of the latter being one of the most fruitful achievements of the Arveson theory for physics. Is the vacuum separating, i.e., being not annihilated by any A, what is typical for thermal vacua, then $Sp^{\tau}(A) = Sp^{U}(A\Omega)$. Thus there is the chance that the spectrum of an exact thermal Green's function displays the whole algebraic spectrum, a question which we shall investigate in the forthcoming paper [13]. On the other hand, the spectral projections, and thus the spectral degenerations, depend essentially on the representation space. An infinite degeneration of a spectral energy value indicates a certain spectral stability (against compact perturbations) and appears typical for a condensate. This feature seems not to be discussed in terms of Green's functions.

Our subsequently presented investigations of the BCS-models illustrate that spectral questions of many body systems are very subtle. For finite temperature representations we find very stable spectral excitations, which are even invariant against arbitrary bounded perturbations. They belong to the so-called Connes spectrum, which is an algebraic invariant of the represented observable algebra. The difference between the energy of the thermal vacuum (mostly renormalized to zero in algebraic quantum field theory) and the first positive term of the Connes spectrum is to be indentified with the gap. The first positive term of the Connes spectrum is, however, not equal to the first positive term of the Arveson spectrum, and it is not at all the lowest excitation, since both spectra are symmetric around zero: The macroscopic thermal vacuum represents an infinite energy reservoir and may be arbitrarily deexcited.

By decomposing the thermal vaccum into an integral over algebraic pure energy states (which as vector states belong to inequivalent representations) we obtain further insights. The decomposing measure provides us with a (temperature dependent) statistics for the relative occupation probabilities of the macroscopic energy values, which fluctuate in thermal equilibrium. These relative occupation probabilities are involved in the energy renormalization (by infinite values). Thus the thermal vacuum obtains its renormalized energy value not by a divergent c-number, but by a divergent operator subtraction. This identifies the deeper, but physically plausible, reason why the renormalized temperature Hamiltonian is not affiliated with the represented observable algebra.

In order to relate the renormalization problem with the Borchers-Arveson theorem, we have expounded the latter in a suitable form in Appendix C. Our analysis demonstrates, that for finite temperatures there is no way to obtain a renormalized Borchers-Arveson Hamiltonian, which would be bounded from below and then would be affiliated with the represented observable algebra. This questions certain procedures in the theory of thermal Green's functions, which work with a lower bounded spectrum. This disclaims also the presumptousness of axiomatic quantum field theory that the temperature Hamiltonian is no observable [14]. Our answer is, that by a renormalization procedure the meaning of an observable is not changed. The implications are alarming: Those theorems of axiomatic quantum field theory, which depend essentially on a too narrow concept of observable algebra, are dubious in regard to their physical significance.

For many body physics it is interesting that our investigation gets across many macroscopically occupied quasi-particle energies. Nevertheless they exhibit no sign of coherence, in whatever sense. Thus the spectral characterization of a condensed state has to be refined. To give a hint, we conclude the present Introduction with a proposal, which illustrates even more the usefulness of the Arveson spectral theory. Recall that also in the case of a symmetry breaking phase transition (of the second kind) the limiting Gibbs states retain their symmetry, being statistical mixtures of the pure phase states. A condensed many body state of this kind may then be characterized by the fact that its Arveson spectral projection has a central part (in the weak closure of the represented observable algebra), commuting with all other observables. This definition applies also to the limiting ground state, which displays less spectral stability than the temperature states for its lower bounded, renormalized Hamiltonian. In physical terms this amounts to the spectral characterization of a situation, where there arises an order parameter, the latter belonging to the center of the represented observable algebra. Observe that the order parameter is as an element of the algebra of an observable, a classical field, and not a macroscopic wave function. Its possible attaining of complex values is a question of notational convenience and may be easily avoided. Therefore its nonlinear dynamical equation does not constitute a breaking of the quantum mechanical superposition principle.

2. Model Assumptions, Limiting Dynamics, and KMS-States

In the BCS-model the effective interactions between the electrons are split into two parts: One part is subsumed into a lattice periodic external potential and gives rise to the Bloch wave functions with energies $\eta_{\vec{k}}$. In a shell around the Fermi energy surface in momentum space one has as second part a pair-pair interaction which is in the average attractive.

The Bloch eigenstates are used to realize the electronic CAR-algebra as a tensor product. Considering a certain numbering of the momenta $\vec{k}$ in a shell around the Fermi surface by $\vec{k} = \vec{k}(k), k \in \mathbb{Z}$, we have for each k two spin values $\sigma \in \{\uparrow', \downarrow\}$. In the sense of a pair formalism we combine $(k, \uparrow)$ with $(-k, \downarrow)$, and the CAR-algebra $\mathfrak{A}$ is written

$$\mathfrak{A} \cong \bigotimes_{k\in\mathbb{N}} \mathfrak{B} \tag{2.1}$$

with $\mathfrak{B} \cong \mathbb{M}_4 \cong \mathbb{M}_2 \otimes \mathbb{M}_2$.

We introduce a *quasi-local* structure in momentum space by associating the local algebra $\mathfrak{A}_\Lambda := \otimes_{k\in\Lambda}\mathfrak{B}$ with each finite subset $\Lambda \in \mathfrak{L} := \{\Lambda \subset \mathbb{N} \mid |\Lambda| < \infty\}$. Dropping the embedding operators we have $\mathfrak{A}_0 := \bigcup_{\Lambda\in\mathfrak{L}} \mathfrak{A}_\Lambda$ as a norm dense sub-algebra of $\mathfrak{A}$.

According to our numbering of the momenta we have the Jordan-Wigner representation for annihilation operators $c_{k,\sigma}$, $k \in \mathbb{N}$ (that is the numbering set of the Bloch momenta), $\sigma \in \{\uparrow, \downarrow\}$

$$\begin{aligned} c_{k\uparrow} &= \Big(\bigotimes_{j=1}^{k-1} (\sigma_z \otimes \sigma_z)\Big) \otimes (\sigma_z \otimes \sigma_-) \otimes \Big(\bigotimes_{j=k+1}^{\infty} \mathbb{1}_4\Big), \\ c_{-k\downarrow} &= \Big(\bigotimes_{j=1}^{k-1} (\sigma_z \otimes \sigma_z)\Big) \otimes (\sigma_- \otimes \mathbb{1}_2) \otimes \Big(\bigotimes_{j=k+1}^{\infty} \mathbb{1}_4\Big), \end{aligned} \tag{2.2}$$

where σ_x, σ_y, σ_z are the Pauli matrices and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$.

The *local Hamiltonian* for a finite set Λ of Bloch modes is obtained by adding to the Bloch energy the pair-pair interaction for which we allow rather arbitrary coupling coefficients with non-trivial dynamical phases and obtain (see e. g. [5–17])

$$H_\Lambda := \sum_{k\in\Lambda} \eta_k \left(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}\right) - \sum_{k,k'\in\Lambda} \frac{g_{kk'}}{|\Lambda|} c^*_{k\uparrow}c^*_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow} \tag{2.3}$$

with $g_{kk'} = \overline{g_{k'k}}$ for all $k, k' \in \mathbb{N}$. The η_k are the reduced values of the kinetic energies for electrons in the state k, that is the kinetic energy measured with respect to the Fermi surface. The *chemical potential* will be fixed throughout the paper and does not appear in the notation.

Introducing the pair annihilation and number operators

$$b_k = c_{-k\downarrow}c_{k\uparrow}, \qquad m_k = c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}, \tag{2.4}$$

we write

$$H_\Lambda = \sum_{k\in\Lambda} \eta_k\, m_k - \sum_{k,k'\in\Lambda} \frac{g_{kk'}}{|\Lambda|} b^*_k b_{k'}. \tag{2.5}$$

Note, that the Hamiltonian consists of matrices in $\mathfrak{B}$, placed onto k-indexed lattice points. To

$$\begin{aligned} &c_{k\uparrow} \in \mathfrak{A} \text{ belongs } \sigma_z \otimes \sigma_- \in \mathfrak{B}, \\ &\text{to } c_{k\downarrow} \text{ belongs } \sigma_- \otimes \mathbb{1}_2 \in \mathfrak{B}, \end{aligned} \tag{2.6}$$

and for $b_k, m_k \in \mathfrak{A}$ there are corresponding matrices $b, m \in \mathfrak{B}$.

As mentioned in the Introduction the basic idea behind our approach is to consider a given inhomogeneous BCS-model as a perturbation of a homogeneous one. The latter is obtained uniquely by averaging the given model data

$$\eta := \lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \eta_k, \quad 0 < g := \lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|^2} \sum_{k,k'\in\Lambda} g_{kk'}, \tag{2.7}$$

and has the local Hamiltonians

$$H^0_\Lambda := \sum_{k\in\Lambda} \eta\, m_k - \sum_{k,k'\in\Lambda} \frac{g}{|\Lambda|} b^*_k b_{k'}, \qquad \Lambda \in \mathfrak{L}. \tag{2.8}$$

As a general assumption of our investigation we assume the validity of (2.7).

In order to arrive at a well behaved perturbation theory one has to require that the perturbations

$$P_\Lambda := H_\Lambda - H^0_\Lambda = \sum_{k\in\Lambda} \delta\eta_k m_k - \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} \delta g_{kk'} b^*_k b_{k'} \tag{2.9}$$

with

$$\delta\eta_k := \eta_k - \eta, \qquad \delta g_{kk'} := g_{kk'} - g \tag{2.10}$$

be "small" in some sense. We stipulate:

2.1. Model Assumption

We say that the BCS-model is in the allowed model class, if the constants (2.10) satisfy the following relations:

$$\begin{aligned} &\lim_{k\to\infty} \delta\eta_k = 0, \\ &\lim_{k'\to\infty} \delta g_{kk'} =: \delta g_k \text{ exists with } \lim_{k\to\infty} \delta g_k = 0 \end{aligned} \tag{2.11}$$

and

$$\lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} \left| \delta g_{kk'} - \delta g_k - \overline{\delta g_{k'}} \right| = 0. \tag{2.12}$$

Here $\lim_{\Lambda\in\mathfrak{L}}$ *denotes the net limit over the index set* $\mathfrak{L}$.

Up to now, we have specified the state-independent features of the BCS-models. The meanfield character is expressed, at this quasi-microscopic stage, by the long range and weakness of the interactions, where both attributes tend to limiting values for increasing Λ. It is a speciality of our approach, that we construct even the limiting meanfield dynamics algebraically, choosing a Hilbert space representation only after the preparation conditions for the many body system have been selected. A basic notion within the algebraic meanfield frame are the (state-independent) dynamical phases

$$\delta\vartheta_k = -\mathrm{Arg}\left(1 + \frac{\delta g_k}{g}\right) = -\mathrm{Arg}\left(g + \delta g_k\right). \tag{2.13}$$

The homogeneous limiting Heisenberg dynamics τ^0 is well established; it does, however, not exist as an automorphism group in $\mathfrak{A}$ but only in an extended observable algebra. We have described in [9] a singular perturbation theory in order to construct the inhomogeneous, reduced limiting dynamics in an extended

C*-algebra, which beside the electron observables contains also classical observables. The latter are indexed by means of states ϱ on the one-cell algebra $\mathfrak{B}$.

2.2. Definition

The (global) classically extended algebra of $\mathfrak{A} = \bigotimes \mathfrak{B}_k$ is defined as the algebraic tensor product, completed in the here unique C-cross norm,*

$$\mathcal{C} := \mathfrak{A} \otimes \mathcal{C}(E_{\mathcal{G}}) = \mathcal{C}(E_{\mathcal{G}}, \mathfrak{A}). \tag{2.14}$$

In reference to previous papers, let us stick to our notation $E_{\mathcal{G}}$ for a convex subset of $\mathfrak{S}(\mathfrak{B})$, which provides the indices for the basic classical observables, determined by an internal symmetry group $\mathcal{G}$, and which is something like a classical phase space. From our present global point of view, however, $E_{\mathcal{G}}$ is all of the states on $\mathfrak{B}$. $\mathcal{C}(E_{\mathcal{G}})$ means the complex continuous functions on $E_{\mathcal{G}}$ and $\mathcal{C}(E_{\mathcal{G}}, \mathfrak{A})$ denotes the $\mathfrak{A}$-valued continuous functions on $E_{\mathcal{G}}$, the latter containing the elements $A = (\varrho \mapsto A(\varrho))$, with $\varrho \in E_{\mathcal{G}}$ and $A(\varrho) \in \mathfrak{A}$. The elements $A = (\varrho \mapsto A(\varrho))$ in $\mathcal{C}_{\mathcal{G}}$ with

$A(\varrho) \in \mathfrak{A}_\Lambda, \quad \forall \varrho \in E_{\mathcal{G}}$,
constitute a C*-subalgebra $\mathcal{C}_\Lambda \subset \mathcal{C}$.

We need the following fact on states in $\mathfrak{S}(\mathcal{C})$:

2.3. Proposition

For each $\omega \in \mathfrak{S}(\mathcal{C})$ there exists a measurable family $\varrho \mapsto \omega^\varrho \in \mathfrak{S}(\mathfrak{A})$, called sector components*, and a measure $d\mu_\omega$ on $E_{\mathcal{G}}$, called* sector distribution*, such that for all $A = (\varrho \mapsto A(\varrho)) \in \mathcal{C}$ one has*

$$\langle \omega \,; A \rangle = \int_{E_{\mathcal{G}}} \langle \omega^\varrho \,; A(\varrho) \rangle \; d\mu_\omega(\varrho).$$

From the microscopic point of view, the classical features connected with $\mathcal{C}$ have to be obtained as limits from the most fundamental quasilocal theory, based on $\mathfrak{A}$ (cf., also [13]).

2.4. Definition

A state $\omega \in \mathfrak{S}(\mathcal{C})$ is called microscopically extended *if*

$$\lim_\Lambda \langle \omega \,; AR_\Lambda \otimes \mathbb{1} \rangle = \langle \omega \,; A \otimes R \rangle$$

for all $A \in \mathfrak{A}$ and all meanfield polynomials $R_\Lambda := R(m_\Lambda(e_1), \ldots, m_\Lambda(e_{16}))$ with the weak limit R in the center of $\mathcal{C}$. Here $m_\Lambda(e_i)$ denotes the average over the lattice region Λ of a basis element $e_i \in \mathfrak{B}, 1 \le i \le 16$.

Stated in words: The expectations of $A \otimes R \in \mathcal{C}$ are approximated by the expectations of the $AR_\Lambda \in \mathfrak{A}$, if the state is microscopically extended. The set of considered states should also be rich enough to separate the classical observables. It is specified as a 'meanfield separating folium' in and is used to define the weaker-than-norm topology for the limiting dynamics.

2.5. Theorem (Inhomogeneous Limiting Dynamics)

Fix a meanfield separating folium $\hat{\mathcal{F}} \subset \mathfrak{S}(\mathcal{C})$ to define a σ-weak topology. For each BCS-model satisfying Assumptions 2.1 there is a unique C^*-dynamical system $(\mathcal{C}, \mathbb{R}, \tau)$ such that for each $A \in \mathcal{C}_\Lambda = \mathcal{C}(E_{\mathcal{G}}, \mathfrak{A}_\Lambda)$ and each $\Lambda \in \mathfrak{L}$ there is a $t_0 > 0$ such that for $0 \le |t| < t_0$ the following limits exist

$$\tau_t(A) = \sigma\text{-weak-}\lim_{\Lambda' \in \mathfrak{L}} (\tau_t^0)^{P_{\Lambda'}}(A) = (\tau_t^0)^{P_\Lambda^\beta}(A),$$

where a superscript like P_Λ^β denotes the perturbation of the automorphism dynamics. Here it is given explicitly by

$$P_\Lambda^\beta = \sum_{k \in \Lambda} \delta h_k \;\in\; \mathcal{C}(E_{\mathcal{G}}, \mathfrak{A}_\Lambda) \subset \mathcal{C}.$$

We use the followoing functions in $\mathcal{C}(E_{\mathcal{G}}, \mathfrak{A}_{\{k\}})$:

$$\begin{aligned} \varrho \to h_k^0(\varrho) := \eta\, m_k - g\,\big(\langle \varrho\,; b\rangle\; b_k^* \\ +\; \langle \varrho\,; b^*\rangle\; b_k \big), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \varrho \to \delta h_k(\varrho) := \delta\eta_k m_k - \delta g_k \,\langle \varrho\,; b\rangle\, b_k^* \\ -\; \overline{\delta g_k}\,\langle \varrho\,; b^*\rangle\, b_k, \end{aligned} \tag{2.16}$$

$$\varrho \to h_k(\varrho) := h_k^0(\varrho) + \delta h_k(\varrho). \tag{2.17}$$

For arbitrary $A \in \mathcal{C}(E_{\mathcal{G}}, \mathfrak{A}) \cong \mathcal{C}$, $\tau_t(A)$ writes as

$$[\tau_t(A)](\varrho) = \big(\bigotimes_{k\in\mathbb{N}} \mathrm{e}^{\mathrm{i}\,t\,h_k(\varrho)}\big)\, A(\gamma_t\varrho)\, \big(\bigotimes_{k\in\mathbb{N}} \mathrm{e}^{-\mathrm{i}\,t\,h_k(\varrho)}\big), \tag{2.18}$$

where the infinite tensor products are defined by their simultaneous local limits in the pointwise norm topology of $\mathcal{C}$. Here γ is the classical flow on $E_{\mathcal{G}}$ generated by h^0, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t\varrho = [h^0(\gamma_t\varrho), \gamma_t\varrho], \quad \varrho \in E_{\mathcal{G}}. \tag{2.19}$$

Denoting

$$\varrho \to H_\Lambda(\varrho) := \sum_{k\in\Lambda} h_k(\varrho) \in \mathcal{C}(E_\mathcal{G}, \mathfrak{A}_\Lambda) \equiv \mathcal{C}_\Lambda, \quad (2.20)$$

the generator L of τ_t has on the core $\mathcal{C}_0^1$ (the continuously differentiable functions in $\mathcal{C}$ with values in $\mathfrak{A}_\Lambda$ for some $\Lambda \in \mathfrak{L}$) the form

$$\begin{aligned}&[L(A)](\varrho) = [H_\Lambda(\varrho), A(\varrho)] - \mathrm{i}\,[\lambda^* A](\varrho),\\ &A \in \mathcal{C}^1(E_\mathcal{G}, \mathfrak{A}_\Lambda)\end{aligned} \quad (2.21)$$

with (the vector field) $[\lambda^* A](\varrho) := \frac{\mathrm{d}}{\mathrm{d}t} A(\gamma_t \varrho)|_{t=0}$ for all $\varrho \in E_\mathcal{G}$.

The fine point in this argumentation is the independence of the C*-automorphism group $\tau \in \mathrm{Aut}(\mathcal{C})$ of the special σ-weak topology.

The classical part γ of the limiting dynamics τ moves the sector indices ϱ. One should not interpret this classical phase space dynamics as a nonlinear Schrödinger dynamics for the 'macroscopic wave function', where the latter arises here as the classical gap-observable (in rough agreement with the Gorkov-treatment [15]) and underlies the Heisenberg dynamics! An equilibrium state $\omega \in \mathfrak{S}(\mathcal{C})$ must have a sector distribution, which is supported by γ-invariant sector indices. This holds especially for the thermodynamic equilibrium and ground states, which are determined via the KMS-condition and the corresponding low temperature limits. The KMS-condition for our model class is equivalent to the self-consistency equations. The latter concern certain parameters of the stationary sector indices. For the BCS-models it is deduced from the model assumptions that the sector indices for the macroscopically pure (that is factorial) equilibrium states are functions of the temperature β and the macroscopic phase angle ϑ (the chemical potential being a fixed model parameter).

2.6. Proposition

Consider a weakly inhomogeneous BCS-model satisfying Assumptions 2.1 and choose a $\beta \in (0, +\infty)$.

(i) *The extremal β-KMS-states $\omega^{\beta\vartheta}$ for the C*-dynamical system $(\mathcal{C}, \mathbb{R}, \tau)$ are locally given by the density operators*

$$\begin{aligned}\varrho_\Lambda^{\beta\vartheta} &= \mathrm{e}^{-\xi_\Lambda - \beta H_\Lambda^{\beta\vartheta}} = \bigotimes_{k\in\Lambda} \mathrm{e}^{-\xi_k - \beta h_k^{\beta\vartheta}}\\ &= \bigotimes_{k\in\Lambda} \varrho_k^{\beta\vartheta}, \quad \Lambda \in \mathfrak{L},\end{aligned} \quad (2.22)$$

where

$$\begin{aligned}H_\Lambda^{\beta\vartheta} := \sum_{k\in\Lambda} \Big\{ &\eta_k m_k - \Delta_k(\beta)[\mathrm{e}^{-\mathrm{i}(\vartheta+\delta\vartheta_k)} b_k^*\\ &+ \mathrm{e}^{\mathrm{i}(\vartheta+\delta\vartheta_k)} b_k] \Big\} =: \sum_{k\in\Lambda} h_k^{\beta\vartheta}.\end{aligned}$$

(ξ_Λ is by the normalization of the density operator the logarithm of the partition function.)

(ii) *The (absolute values of the) gaps $\Delta_k(\beta) \equiv \Delta_k$, appearing in $H_\Lambda^{\beta\vartheta}$, are determined by the selfconsistency equations [9]*

$$\lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k\in\Lambda} g_{lk} \frac{\Delta_k}{2E_k} \mathrm{e}^{-\mathrm{i}\vartheta_k} \tanh\left(\frac{\beta E_k}{2}\right) = \Delta_l\, \mathrm{e}^{-\mathrm{i}\vartheta_l}, \quad (2.23)$$

where we have introduced the state-dependent energy values

$$E_k = E_k(\beta) = \sqrt{\eta_k^2 + \Delta_k(\beta)^2}. \quad (2.24)$$

All solutions of (2.23) have the form

$$\Delta_k(\beta) := \left|1 + \frac{\delta g_k}{g}\right| \left| g \left\langle \varrho_0^{\beta\vartheta}; b \right\rangle \right| = \left|1 + \frac{\delta g_k}{g}\right| \Delta_0(\beta),$$

where $\Delta_0(\beta)$ is the absolute homogeneous gap, determined by an algebraic equation, which has a nontrivial solution for $\beta > \beta_c$. This determines also the homogeneous one-lattice-point density operator

$$\begin{aligned}\varrho_0^{\beta\vartheta} = \exp\{&-\xi_0(\beta) - \beta[\eta m - \Delta_0(\beta)\\ &\cdot [\mathrm{e}^{-\mathrm{i}\vartheta} b^* + \mathrm{e}^{\mathrm{i}\vartheta} b]]\} \in \mathfrak{S}(\mathfrak{B}).\end{aligned} \quad (2.25)$$

That is, we have – for each model of the class – solutions of the selfconsistency equation, which for β larger than the critical β_c are nontrivial. The phase angles of the nontrivial complex gap values $\mathrm{e}^{-\mathrm{i}(\vartheta+\delta\vartheta_k)}\Delta_k(\beta)$ decompose into the macroscopic phase ϑ plus the dynamical, microscopic phase fluctuations $\delta\vartheta_k$ (2.13).

(iii) *The average of the pure phase states, formed over the phase angle interval $\vartheta \in [0, 2\pi) =: I$,*

$$\omega^\beta = \int_I \omega^{\beta\vartheta}\, \delta\,\vartheta/2\pi \quad (2.26)$$

(nontrivial integration for $\beta > \beta_c$) is the unique gauge invariant β-KMS state in terms of its central decomposition (i.e., in its decomposition into pure phase states).

With the reasoning of [10] one concludes that ω^{β} is the unique limiting Gibbs state for the inhomogeneous model, a remarkable result. Its decomposition (2.26) expresses the finest filtering, which is possible by means of classical observables and is supported by the pure phase states $\omega^{\beta\vartheta}$. Mathematically this central decomposition defines a special orthogonal measure on $\mathfrak{S}(\mathcal{C})$ and – by restricting the observables – on $\mathfrak{S}(\mathfrak{A})$. We perform the spectral analysis of the limiting dynamics only in the GNS-representations $(\Pi_{\beta\vartheta}, \mathcal{H}_{\beta\vartheta}, \Omega_{\beta\vartheta})$ over the pure phase states $\omega^{\beta\vartheta}$ and in their low temperature limits.

The physically appropriate observable algebras, corresponding to these representations, are the weak closure von Neumann algebras $\mathfrak{M}_{\beta\vartheta} = \Pi_{\beta\vartheta}(\mathcal{C})'' = \Pi_{\beta\vartheta}(\mathfrak{A})''$. Observe that, because of the product structure of the original electron algebra $\mathfrak{A}$ and of the pure phase states, one has the von Neumann incomplete tensor product

$$\mathfrak{M}_{\beta\vartheta} = \bigotimes_{k\in\mathbb{N}}^{\omega^{\beta\vartheta}} \mathfrak{B}_k$$

(cf., also Bures [16]).

In the GNS-representation $(\Pi_{\beta\vartheta}, \mathcal{H}_{\beta\vartheta}, \Omega_{\beta\vartheta})$ the scalar products are evaluated in terms of the state $\omega^{\beta\vartheta}$, which fixes the sector index onto $\varrho = \varrho_0^{\beta\vartheta}$. The global automorphism-dynamics of Theorem 2.5, placed into this representation, also sees only this time invariant sector index (2.25). The automorphisms are weakly extensible and we obtain:

2.7. Proposition (Inhomogeneous Pure Phase Dynamics)

For each BCS-model satisfying Assumptions 2.1 and for each $\beta > \beta_c$ and $\vartheta \in I$, the pure phase component of the limiting dynamics is given by the W^*-dynamical system $(\mathfrak{M}_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta})$ acting on $A \in \mathfrak{M}_{\beta\vartheta}$ as

$$\tau_t^{\beta\vartheta}(A) = \Big(\bigotimes_{k\in\mathbb{N}} \mathrm{e}^{\mathrm{i}\,t\,h_k^{\beta\vartheta}}\Big)\, A\, \Big(\bigotimes_{k\in\mathbb{N}} \mathrm{e}^{-\mathrm{i}\,t\,h_k^{\beta\vartheta}}\Big), \quad (2.27)$$

where the product operator is meant in the sense of von Neumann [17], resp. Bures [16].

The local Heisenberg generator has the form

$$L^{\beta\vartheta}(A) = \frac{\mathrm{d}}{\mathrm{id}t}\tau_t^{\beta\vartheta}(A)|_{t=0} = [H_\Lambda^{\beta\vartheta}, A], \quad A \in \Pi_{\beta\vartheta}(\mathcal{C}_\Lambda). \quad (2.28)$$

3. Spectral Properties of Pure Phase and Ground State Dynamics

Let us state again that the reduced limiting dynamics, given in the Heisenberg picture by the C*-dynamical system $(\mathcal{C}, \mathbb{R}, \tau)$ for the classically extended observable algebra $\mathcal{C}$, represents the global point of view. The classical variables in the center of $\mathcal{C}$ play physically the role of order parameters, assuming, in general, time-dependent values. In many body physics one is mostly concerned with situations, where the order parameters display fixed, stationary values. This situation is mathematically characterized by a time invariant factor representation of $\mathcal{C}$ (and of $\mathfrak{A}$). The microscopic energy concept refers to such type of situation. The physically relevant energy values are obtained by the action of τ in the representation and its extension to the representation von Neumann algebra. This energy concept obviously depends on the representation, especially on the temperature.

In our model discussion we have, therefore, to look for the GNS-representations over the extremal KMS-states and their low temperature limits. For finite temperatures we have already introduced the W*-dynamical systems $(\mathfrak{M}_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta})$. As is not difficult to demonstrate the low temperature limits exist for the pure phase states, leading to well defined pure ground states $\omega^{\infty\vartheta}$ for $(\mathcal{C}, \mathbb{R}, \tau)$. The corresponding W*-dynamical system $(\mathfrak{M}_{\infty\vartheta}, \mathbb{R}, \tau^{\infty\vartheta})$ is constructed quite analogously to the finite temperature case. For determining the spectra of the mentioned W*-dynamical systems we perform the Bogoliubov-Valatin transformation as a $*$-automorphism in $\mathfrak{A} \subset \mathfrak{M}_{\beta\vartheta}$, identifying $\mathfrak{A}$ here with its faithful GNS-representation for arbitrary but fixed $\beta \in (\beta_c, +\infty], \vartheta \in I$.

Altogether we communicate the following results, which cover the finite temperature and the groundstate situation and constitute a sharpening of the usual BCS-treatment in theoretical physics. (We omit frequently the indices $\beta\vartheta$, when no confusion is likely to arise,)

3.1. Proposition

(i) *The prescription*

$$\begin{aligned}\gamma_{k0} \equiv \gamma_{k0}(\beta,\vartheta) &:= \chi^{\beta\vartheta}(c_{k\uparrow}) \\ &= u_k c_{k\uparrow} - v_k \mathrm{e}^{-\mathrm{i}(\vartheta+\delta\vartheta_k)} c^*_{-k\downarrow},\end{aligned} \quad (3.1)$$

$$\begin{aligned}\gamma_{k1} \equiv \gamma_{k1}(\beta,\vartheta) &:= \chi^{\beta\vartheta}(c_{-k\downarrow}) \\ &= v_k \mathrm{e}^{-\mathrm{i}(\vartheta+\delta\vartheta_k)} c^*_{k\uparrow} + u_k c_{-k\downarrow}\end{aligned} \quad (3.2)$$

with

$$u_k(\beta) := \frac{1}{\sqrt{2}}\sqrt{1+\frac{\eta_k}{E_k(\beta)}},$$
$$v_k(\beta) := \frac{1}{\sqrt{2}}\sqrt{1-\frac{\eta_k}{E_k(\beta)}}$$

gives by $*$*-algebraic and norm-continuous extension a* $*$*-isomorphism* $\chi^{\beta\vartheta} \in Aut(\mathfrak{A})$ *with*

$$\chi^{\beta\vartheta}(\mathfrak{A}_\Lambda) = \mathfrak{A}_\Lambda, \quad \forall\Lambda \in \mathfrak{L}.$$

(The Bogoliubov transformation $\chi^{\beta\vartheta}$ *is not* σ*-weakly continuous, a feature which is connected with the fact that electron pairs, but not quasi-particle pairs, condense, cf. also the end of Section 6.)*

(ii) *The so-called 'model Hamiltonian' [6] in written terms of the* γ*-operators*

$$H_\Lambda^{\beta\vartheta} = \sum_{k\in\Lambda} E_k(\gamma_{k0}^*\gamma_{k0}+\gamma_{k1}^*\gamma_{k1}) - \sum_{k\in\Lambda}(E_k-\eta_k)\mathbb{1},$$
$$\forall\Lambda \in \mathfrak{L}, \tag{3.3}$$

and thus the local restriction of the limiting pure phase equilibrium state has the form

$$\langle\omega^{\beta\vartheta}; A\rangle =$$
$$\mathrm{tr}_\Lambda[\exp(-\xi_\Lambda - \beta\sum_{k\in\Lambda}E_k(\gamma_{k0}^*\gamma_{k0}+\gamma_{k1}^*\gamma_{k1}))A], \tag{3.4}$$
$\forall A \in \mathfrak{A}_\Lambda$ and $\forall\Lambda \in \mathfrak{L}$,

where ξ_Λ *is determined by normalization (that is the logarithm of the local partition function after having performed the thermodynamic limit).*

(iii) *There exist the low temperature limits*

$$w^* - \lim_{\beta\to\infty}\omega^{\beta\vartheta} = \omega^{\infty\vartheta}, \tag{3.5}$$

where the pure ground states are given in terms of their application to $A = (\varrho \to A(\varrho)) \in \mathcal{C}_\Lambda$ *by*

$$\langle\omega^{\infty\vartheta}; A\rangle = tr_\Lambda[\bigotimes_{\kappa\in\Lambda} n_\kappa^\perp(\beta,\vartheta)A(\varrho_0^{\infty\vartheta})],$$
$$\forall\Lambda \in \mathcal{K}. \tag{3.6}$$

Here we have employed the quasi-particle occupation operators

$$n_{k\lambda}(\beta,\vartheta) := \gamma_{k\lambda}^*\gamma_{k\lambda},$$
$$n_{k\lambda}^\perp(\beta,\vartheta) := \mathbb{1} - n_{k\lambda}, \tag{3.7}$$

and introduced the new numbering

$$(k,\lambda) =: \kappa \in \mathcal{K},\ \lambda = 0,1. \tag{3.8}$$

We need the quasi-particle operators in $\mathfrak{B}$*, defined by, cf. (2.6),*

$$\gamma_\uparrow(\beta,\vartheta) := \chi^{\beta\vartheta}(c_\uparrow) \in \mathfrak{B},$$
$$\gamma_\downarrow(\beta,\vartheta) := \chi^{\beta\vartheta}(c_\downarrow) \in \mathfrak{B}.$$

Note that we have used only in $\mathfrak{B}$ *the arrow-index to indicate the kind of the quasi-particle! We have also used for the sector index of the groundstate, given by a state on the* $\mathfrak{B}$*-algebra, the notation*

$$\varrho_0^{\infty\vartheta} = n_\uparrow^\perp(\infty,\vartheta) \otimes n_\downarrow^\perp(\infty,\vartheta), \tag{3.9}$$

remembering the (β,ϑ)*-dependence of the* γ*-operators sporadically.*

The ground states $\omega^{\infty\vartheta}$ *are invariant states of the global dynamical system* $(\mathcal{C}_\mathcal{G}, \mathbb{R}, \tau)$*.*

Note, that from now on the previous pairs $(k,0)$ resp. $(k,1), k \in \mathbb{N}$ are numbered by the discrete indices κ from the denumberable, totally ordered set $\mathcal{K}$, such that the second of the above pairs is the successor of the first. By abuse of notation, the finite subsets of $\mathcal{K}$ are still denoted by Λ, and the set of all Λ is named again $\mathfrak{L}$.

In order to use the Arveson spectral theory for the W*-dynamical systems $(\mathfrak{M}_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta})$, $\beta \in (\beta_c, +\infty]$, $\vartheta \in I$, we employ the notions of the Appendix. An eigenelement $V \in \mathfrak{M}_{\beta\vartheta}$ for, e.g., the pure-phase reduced dynamics, satisfies by definition the relation

$$\tau_t^{\beta\vartheta}(V) = \exp(\mathrm{i}Et)V,\ \forall t \in \mathbb{R},$$

where then $E \in \hat{\mathbb{R}}$, from the dual group of $\mathbb{R} \cong \hat{\mathbb{R}}$ is called the corresponding 'eigenvalue'.

3.2. Definition

(i) *We introduce the following sets of occupation configurations* ε*:*

$$\mathcal{E}_\Lambda := \{\varepsilon : \Lambda \longrightarrow \{1,\perp\}\}; \tag{3.10}$$

$$\mathcal{E} := \{\varepsilon : \mathcal{K} \longrightarrow \{1,\perp\}\}; \tag{3.11}$$

$$r_\Lambda := \mathcal{E} \to \mathcal{E}_\Lambda \text{ restriction of the maps.} \tag{3.12}$$

The algebraic projection onto a specified, local occupation configuration, given by $\varepsilon \in \mathcal{E}$ *is*

$$N_\Lambda^\varepsilon := \bigotimes_{\kappa\in\Lambda} n_\kappa^{\varepsilon_\kappa}, \quad \varepsilon \in \mathcal{E}. \tag{3.13}$$

(ii) *For given* $\varepsilon \in \mathcal{E}$ *we decompose the lattice as* $\mathcal{K} = \mathcal{K}_+^\varepsilon \cup \mathcal{K}_-^\varepsilon$*, where the subsets are defined by*

$$K_+^\varepsilon := \{\kappa \in K | \varepsilon_\kappa = \perp\} \quad \text{occupied operator modes;}$$

$$K_-^\varepsilon := \{\kappa \in K | \varepsilon_\kappa = 1\} \ \text{unoccupied operator modes.}$$

For given $\varepsilon \in \mathcal{E}$ *we define for* $\Lambda \in \mathfrak{L}$ *the following sets*

$$\Lambda_{+/-}^\varepsilon := \Lambda \cap K_{+/-}^\varepsilon$$

so that $\Lambda = \Lambda_+^\varepsilon \,\dot\cup\, \Lambda_-^\varepsilon$.

(iii) *We now introduce the operators*

$$\Gamma_\Lambda^{\varepsilon *} := \prod_{\kappa\in\Lambda_+^\varepsilon} \gamma_\kappa^* \prod_{\kappa\in\Lambda_-^\varepsilon} \gamma_\kappa \ \in \mathfrak{A}_\Lambda \subset \mathfrak{M}_{\beta\vartheta} \tag{3.14}$$

and the energy values

$$E_\Lambda^\varepsilon := \sum_{\kappa\in\Lambda_+^\varepsilon} E_\kappa - \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa, \tag{3.15}$$

where $E_\kappa = E_\kappa(\beta)$ *is from (2.24).*

(iv) *From (3.6) one sees that a pure ground state* $\omega^{\infty\vartheta}$ *is associated with a special operator occupation configuration, denoted by*

$$\mathcal{E} \ni \varepsilon^0 := (\varepsilon_\kappa = \perp, \forall\kappa \in \mathcal{K}),$$
$$\textit{implying } \mathcal{K}_+^{\varepsilon^0} = \mathcal{K},\ E_\Lambda^{\varepsilon^0} = \sum_{\kappa\in\Lambda} E_\kappa.$$

(Note the contradistinction between operator and state occupation configurations!)

In this modified notation we may use the results of [9] to reveal an interesting connection between the stable spectral values and subgroups resp. subsemigroups of $\mathbb{R}$. Observe that the quasi-particle energy of the homogeneous model is $E_0(\beta) = \sqrt{\eta^2 + \Delta_0(\beta)^2}$.

3.3. Proposition

Consider a BCS-model of the allowed class. Let $(\mathfrak{M}_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta})$, $\beta_c < \beta \le +\infty$*, be the W*-dynamical system associated with the GNS-representation over the pure phase KMS- resp. ground state* $\omega^{\beta\vartheta} \in \mathfrak{S}(\mathcal{C})$*. Then it holds:*

(i) *The set* $\{\Gamma_\Lambda^{\varepsilon *} N_{\Lambda'}^{\varepsilon'} | \varepsilon, \varepsilon' \in \mathcal{E},\ \Lambda, \Lambda' \in \mathfrak{L}\}$ *is a* σ*-weakly total set of eigenelements for* $\tau^{\beta\vartheta}$ *with the eigenvalues* E_Λ^ε *depending on the first factor only. (For* $\Lambda = \emptyset$ *the elements* $\Gamma_\Lambda^{\varepsilon *}$ *and* N_Λ^ε *are* $\mathbb{1}$ *and* $E_\Lambda^\varepsilon = 0$*, for all* $\varepsilon \in \mathcal{E}$*.) Thus, by the closedness of the Arveson spectrum, we have*

$$Sp(\tau^{\beta\vartheta}) = \overline{\{E_\Lambda^\varepsilon | \varepsilon \in \mathcal{E},\ \Lambda \in \mathfrak{L}\}}.$$

(ii) *Each of the above eigenvalues is infinitely degenerate, that is, the spectral subspaces* $\mathfrak{M}(\tau^{\beta\vartheta}, \{E\})$, E *an eigenvalue, are infinite dimensional.*

(iii) $Sp(\tau^{\beta\vartheta})$ *is in general no subgroup of* $\mathbb{R}$.

(iv) *The homogeneous spectrum* $E_0(\beta)\mathbb{Z}$ *is contained in* $Sp(\tau^{\beta\vartheta})$ *for all BCS-models in the considered class.*

(v) *For* $E \in Sp(\tau^{\beta\vartheta})$ *and* $n \in \mathbb{Z}$*, the spectral module property (with "+" as the product) follows*

$$E + nE_0(\beta) \in Sp(\tau^{\beta\vartheta}).$$

(vi) *For* $\beta \in (\beta_c, \infty)$ *each von Neumann algebra* $\mathfrak{M}_{\beta\vartheta}$ *is a Connes factor of type* $III_{\lambda(\beta)}$ *with* $\lambda(\beta) = \exp(-\beta E_0(\beta))$*, whereas each* $\mathfrak{M}_{\infty\vartheta}$ *is a factor of type* I_∞.

Let us now turn to the implementing dynamical Hilbert space operators!

We define the time translation operators, implementing the dynamics in the GNS-representation, in the standard way

$$U_t^{\beta\vartheta} \Pi_{\beta\vartheta}(C)\Omega_{\beta\vartheta} = \Pi_{\beta\vartheta}(\tau_t(C))\Omega_{\beta\vartheta}, \quad \forall C \in \mathcal{C}, \tag{3.16}$$

and extend them to unitary operators in $\mathcal{H}_{\beta\vartheta}$ (retaining their symbolic notation). By construction the $U_t^{\beta\vartheta}$ leave the vector $\Omega_{\beta\vartheta}$ invariant, a feature which uniquely characterizes this unitary implementation. The explicit formula for the extended unitaries will be given in (5.4). The canonically associated Hamiltonians are

$$K^{\beta\vartheta} := \frac{\mathrm{d}}{\mathrm{d}t} U^{\beta\vartheta}|_{t=0}, \ \text{with } K^{\beta\vartheta}\Omega_{\beta\vartheta} = 0. \tag{3.17}$$

Let us call them *GNS-Hamiltonians* and discuss their eigenvectors and their (operator) spectra, denoted $\sigma(K^{\beta\vartheta})$.

3.4. Proposition

(i) *For* $\beta \in (\beta_c, +\infty)$ *the set*

$$\{\Gamma_\Lambda^{\varepsilon *} N_{\Lambda'}^{\varepsilon'} \Omega_{\beta,\vartheta} \mid \varepsilon, \varepsilon' \in \mathcal{E},\ \Lambda, \Lambda' \in \mathfrak{L}\} \tag{3.18}$$

is a total set of eigenvectors for $K^{\beta\vartheta}$.

(ii) *The operator spectrum of the GNS-Hamiltonians is for* $\beta \in (\beta_c, +\infty)$ *given as*

$$\sigma(K^{\beta\vartheta}) = Sp(\tau^{\beta\vartheta}) = \overline{\{E_\Lambda^\varepsilon | \varepsilon \in \mathcal{E},\ \Lambda \in \mathfrak{L}\}}, \tag{3.19}$$

where the first equality follows from the separability of the cyclic vector according to Appendix B. Thus, as for the Heisenberg spectrum, the stable spectral values of the operator spectrum form the group $\mathbb{Z}E_0(\beta)$ *and the module property holds also for this spectrum with respect to this group.*

(iii) *For* $\beta = \infty$ *the set*

$$\{\Gamma_\Lambda^{\varepsilon^0 *} \Omega_{\infty\vartheta} \mid \Lambda \in \mathfrak{L}\} \tag{3.20}$$

is a total set of eigenvectors for $K^{\infty\vartheta}$.

(iv) *The operator spectrum of the GNS-Hamiltonians, for* $\beta = \infty$, *is*

$$\sigma(K^{\infty\vartheta}) = \overline{\{E_\Lambda^{\varepsilon^0} \mid \Lambda \in \mathfrak{L}\}}\ (\neq Sp(\tau^{\infty\vartheta})). \tag{3.21}$$

Thus the spectrum is non-negative.

(v) *Each of the above eigenvalues is infinitely degenerate.*

(vi) $\sigma(K^{\infty\vartheta})$, $\vartheta \in I$, *are in general no sub-semigroups of* $\mathbb{R}_+$.

(vii) *Denote by* $\mathbb{N}_0$ *the additive semigroup* $\mathbb{N}\cup\{0\} \subset \mathbb{R}_+$. *Then* $E_0\mathbb{N}_0 \subset \sigma(K^{\infty\vartheta})$, $\vartheta \in I$, *for all BCS-models in the considered class.*

(viii) *For* $E \in \sigma(K^{\infty\vartheta})$, $\vartheta \in I$, *and* $n \in \mathbb{N}_0$ *it follows*

$$E + nE_0(\infty) \in \sigma(K^{\infty,\vartheta}).$$

We see that the algebraic Heisenberg spectrum is "halved" by going over to the operator spectrum in the groundstate representation. Remarkable is the stability of the homogeneous ground state spectrum, inspite of the Connes spectrum being trivial for the ground state von Neumann algebra. This illustrates that the Connes spectrum is not sufficient for discussing the physically stable spectral values (and by no means for discussing the total spectrum).

Since only the stable energies are macroscopically measurable and the measurement of the gap is performed by macroscopic devices, the theoretical definition of the gap should include some stability requirements. In our idealized model a natural stability is displayed by the homogeneous spectrum for all $\beta \in (\beta_c, +\infty]$.

Since the unitary implementations of the respective Heisenberg dynamics may be altered by multiplying a unitary operator from the commutant, we have especially in the reducible temperature representation a vast variety of implementing Hamiltonians. For analyzing these possibilities, we shall decompose the temperature state into an integral over pure energy states, the latter allowing only c-number renormalizations – may be, with singular constants – for the corresponding Hamiltonians.

4. Renormalized Hamiltonians for Pure Energy States

We discuss first the decomposition of $\omega^{\beta\vartheta}, \beta \in (\beta_c, \infty), \vartheta \in [0, 2\pi)$ – the extremal KMS-states of $(\mathcal{C}, \mathbb{R}, \tau)$ – into pure states with sharp quasi-particle energies and then treat the energy renormalization in the representations over these pure energy states. These are more general but similar to the ground state representations. In the next Section the results are pieced together in order to analyze the energy renormalization in the finite temperature representation.

As mentioned in Section 2, the state $\omega^{\beta\vartheta}$ sees only the $\varrho_0^{\beta\vartheta}$-sector of $\mathcal{C}$. If $\omega^{\beta\vartheta}$ is decomposed as a state on $\mathfrak{A}$, then this decomposition may be lifted to a decomposition in $\mathfrak{S}(\mathcal{C})$ by assigning each component state of the $\mathfrak{A}$-decomposition the sharp sector index $\varrho_0^{\beta\vartheta}$. Thus it is sufficient to perform the decomposition of $\omega^{\beta\vartheta}$ in $\mathfrak{S}(\mathfrak{A})$. We construct, in what follows, a special decomposition of $\omega^{\beta\vartheta}$, using its GNS-representation $(\Pi_{\beta\vartheta}, \mathcal{H}_{\beta\vartheta}, \Omega_{\beta\vartheta})$.

Since the BCS-Hamiltonians are diagonal in the quasi-particle operators, the quasi-particle occupation number operators $N_\Lambda^\varepsilon = \bigotimes_{\kappa\in\Lambda} n_\kappa^{\varepsilon_\kappa}$ project onto energy eigen-elements. The projected (reduced) cyclic state vector of the finite-temperature representation, arising after a measurement of all of the quasi-particle energies with index $\kappa \in \Lambda$, is

$$\Omega_\varepsilon^\Lambda := \frac{N_\Lambda^\varepsilon \Omega_{\beta\vartheta}}{w_\Lambda^\varepsilon} \tag{4.1}$$

with w^ε_Λ positive, such that

$$g^\varepsilon_\Lambda := (w^\varepsilon_\Lambda)^2 := (\Omega_{\beta\vartheta}|N^\varepsilon_\Lambda \Omega_{\beta\vartheta}) = \prod_{\kappa\in\Lambda} g^{\varepsilon_\kappa}_\kappa, \quad (4.2)$$

$$\text{where} \quad g^{\varepsilon_\kappa}_\kappa \,(= g^{\varepsilon_\kappa}_\kappa(\beta)) := (\Omega_{\beta\vartheta}|n^{\varepsilon_\kappa}_\kappa \Omega_{\beta\vartheta}) \quad (4.3)$$

$$= \begin{cases} \exp(-\beta E_\kappa)/[1+\exp(-\beta E_\kappa)], & \varepsilon_\kappa = 1, \\ 1/[1+\exp(-\beta E_\kappa)], & \varepsilon_\kappa = \perp. \end{cases} \quad (4.4)$$

The normal state on $\mathfrak{M}_{\beta\vartheta} = \Pi_{\beta\vartheta}(\mathcal{C})''$, given by $\Omega^\Lambda_\varepsilon$ is denoted by $\omega^\varepsilon_\Lambda$, being simultaneously a state on $\mathfrak{A} \subset \mathfrak{M}_{\beta\vartheta}$. We may, however, interpret $\omega^\varepsilon_\Lambda$ also as a state on $\mathcal{C}$, in which case we denote it by $\omega^\varepsilon_\Lambda(\varrho^{\beta\vartheta}_0)$.

In order to obtain the corresponding energy (resp. particle) projection operators in the commutant of the von Neumann algebra $\mathfrak{M}_{\beta\vartheta}$, with the cyclic and separating vector $\Omega_{\beta\vartheta}$, we use the fundamental concepts of the Tomita-Takesaki theory [11, 18, 19]. The latter begins usually with the introduction of the antilinear operator $S_0 A\Omega_{\beta\vartheta} := A^*\Omega_{\beta\vartheta}, A \in \mathfrak{M}_{\beta\vartheta}$, and the polar decomposition $S = J\Delta^{1/2}$ of its selfadjoint extension. Here J is the 'modular conjugation', an involutive mapping in $\mathcal{H}_{\beta\vartheta}$, with $J^2 = \mathbb{1}, J^* = J$ and $J\Omega_{\beta\vartheta} = \Omega_{\beta\vartheta}$. The positive selfadjoint operator Δ is the 'modular operator'. The mapping

$$j(A) := JAJ, \ \forall A \in \mathfrak{M}_{\beta\vartheta},$$

defines an anti-linear $*$-isomorphism from $\mathfrak{M}_{\beta\vartheta}$ onto the commutant $\mathfrak{M}'_{\beta\vartheta}$ (which 'doubles the operators' in the physical jargon of thermal field theory [8].

4.1. Definition

By means of the anti-linear $*$-isomorphism j we map the energy projection operators into the commutant $\mathfrak{M}'_{\beta\vartheta}$:

$$\widetilde{N}^\varepsilon_\Lambda := j(N^\varepsilon_\Lambda), \quad \forall \varepsilon \in \mathcal{E}, \forall \Lambda \in \mathfrak{L}. \quad (4.5)$$

They generate the following commutative von Neumann algebras (where the join $\vee$ designates the smallest von Neumann algebra containing the given quantities)

$$\widetilde{\mathcal{N}}_\Lambda := \vee\{j(N^\varepsilon_\Lambda)|\ \varepsilon \in \mathcal{E}\}. \quad (4.6)$$

$$\widetilde{\mathcal{N}} := \vee\{\widetilde{\mathcal{N}}_\Lambda|\ \Lambda \in \mathfrak{L}\}. \quad (4.7)$$

It holds

$$\mathbb{1} = \sum_{r_\Lambda(\varepsilon)\in\mathcal{E}_\Lambda} \widetilde{N}^\varepsilon_\Lambda, \ \widetilde{N}^\varepsilon_\Lambda \cdot \widetilde{N}^{\varepsilon'}_\Lambda = \delta_{r_\Lambda(\varepsilon), r_\Lambda(\varepsilon')} \widetilde{N}^\varepsilon_\Lambda, \quad \varepsilon, \varepsilon' \in \mathcal{E},$$

where summation over $r_\Lambda(\varepsilon)$ counts the $r_\Lambda(\varepsilon) \in \mathcal{E}_\Lambda$, and not the $\varepsilon \in \mathcal{E}$. Observe, that also

$$g^\varepsilon_\Lambda = (\Omega_{\beta\vartheta}|\ \widetilde{N}^\varepsilon_\Lambda\, \Omega_{\beta\vartheta}), \quad \forall \varepsilon \in \mathcal{E}.$$

We associate now, using the general theory of orthogonal measures, with $\widetilde{\mathcal{N}}_\Lambda$ the measure $\widetilde{\mu}_\Lambda$ and with $\widetilde{\mathcal{N}}$ the measure $\widetilde{\mu}$ on the state space $\mathfrak{S}(\mathfrak{A})$. That means in more explicit terms

$$\widetilde{\mu}_\Lambda = \sum_{r_\Lambda(\varepsilon)\in\mathcal{E}_\Lambda} g^\varepsilon_\Lambda\, \delta_{\omega^\varepsilon_\Lambda}, \quad \widetilde{\mu} = w^* - \lim_\Lambda \widetilde{\mu}_\Lambda, \quad (4.8)$$

where δ_ω denotes the Dirac measure at the point ω. We have the barycentric decompositions

$$\begin{aligned} \omega^{\beta\vartheta} &= \sum_{r_\Lambda(\varepsilon)\in\mathcal{E}_\Lambda} g^\varepsilon_\Lambda \omega^\varepsilon_\Lambda \\ &= \int_{\mathfrak{S}(\mathfrak{A})} \omega \ \mathrm{d}\widetilde{\mu}_\Lambda(\omega), \ \forall \Lambda \in \mathfrak{L}, \end{aligned} \quad (4.9)$$

$$\omega^{\beta\vartheta} = \int_{\mathfrak{S}(\mathfrak{A})} \omega \ \mathrm{d}\widetilde{\mu}(\omega). \quad (4.10)$$

With increasing Λ the energy filtering of $\omega^{\beta\vartheta}$ becomes finer and finer, and for $\Lambda \subset \Lambda'$ it holds $\widetilde{\mu}_\Lambda \prec \widetilde{\mu}_{\Lambda'}$ in the sense of the Choquet theory (in the present context [11] is sufficient).

Since $\widetilde{\mathcal{N}}$ is generated by tensor products of one-dimensional projections, in each factor of the product, it is maximal abelian and corresponds to the finest possible energy filtering.

Note that $\Omega^\Lambda_\varepsilon \in \mathcal{H}_{\beta\vartheta}$ but the limit $\lim_\Lambda \Omega^\Lambda_\varepsilon$ is not defined in $\mathcal{H}_{\beta\vartheta}$. Nevertheless, we have the limit as a state ω^ε in the algebraic sense. We need the limit as a state in $\mathfrak{S}(\mathcal{C})$:

$$\omega^\varepsilon \equiv \omega^\varepsilon(\varrho^{\beta\vartheta}_0) := \mathrm{w}^* - \lim_\Lambda \omega^\varepsilon_\Lambda(\varrho^{\beta\vartheta}_0). \quad (4.11)$$

That is

$$\left\langle \omega^\varepsilon(\varrho^{\beta\vartheta}_0)\,; A\right\rangle = (\Omega^\Lambda_\varepsilon|A(\varrho^{\beta\vartheta}_0)\Omega^\Lambda_\varepsilon) = \left\langle \omega^\varepsilon_\Lambda(\varrho^{\beta\vartheta}_0)\,; A\right\rangle,$$

$A \in \mathcal{C}_\Lambda$.

It holds that the support of $\widetilde{\mu}$ is defined in the strict sense and is contained in $\{\omega^\varepsilon \mid \varepsilon \in \mathcal{E}\}$. The latter is a continuous standard Borel space if one employs the induced w^*-topology and is obviously Borel isomorphic to $\mathcal{E}$ with the smallest σ-algebra $\Sigma(\mathcal{E})$, containing the cylinder sets $\mathcal{Z}_B(\Lambda) = \{\varepsilon \in \mathcal{E} \,|\, r_\Lambda(\varepsilon) \in B\}$, where

$B \subset \mathcal{E}_\Lambda$. (It is also isomorphic to $[0,1]$ with the real Borel subsets.) The family $\varepsilon \mapsto \omega^\varepsilon$ defines a measurable map of $\mathcal{E}$ into $\mathfrak{S}(\mathcal{C})$. Let us denote by

$$\mu \text{ the tranfer of the } \mathfrak{S}(\mathfrak{A})\text{-measure } \widetilde{\mu} \text{ onto } \mathcal{E}. \tag{4.12}$$

Directly characterized, μ is the unique product measure on $\Sigma(\mathcal{E})$ with

$$\mu(\mathcal{Z}_B(\Lambda)) := \sum_{r_\Lambda(\varepsilon)\in B} g^\varepsilon_\Lambda. \tag{4.13}$$

Transferred to the parameter space $\mathcal{E}$ the Tomita map, associated with the considered orthogonal measure, is a map

$$\Theta_\mu : L^\infty(\mathcal{E},\mu) \longrightarrow \Pi_{\beta\vartheta}(\mathcal{C})', \tag{4.14}$$

uniquely determined by

$$(\Omega_{\beta\vartheta} \mid \Theta_\mu(f)\Pi_{\beta\vartheta}(A)\Omega_{\beta\vartheta}) = \int_{\mathcal{E}} f(\varepsilon)\,\langle\omega^\varepsilon ; A\rangle\,\mathrm{d}\mu(\varepsilon), \tag{4.15}$$

for all $f \in L^\infty(\mathcal{E},\mu)$ and $A \in \mathcal{C}$.

Since we started with the commutative von Neumann algebra in the commutant, we need the inverse Tomita map, which gives, e.g., for $j(M), M \in \mathcal{N}$,

$$[\Theta_\mu^{-1}(j(M))](\varepsilon) = \langle\omega^\varepsilon ; M\rangle\,, \quad \varepsilon \in \mathcal{E}. \tag{4.16}$$

We have by the standard (spatial) decomposition theory [11]:

4.2. Observation

(i) *The energy filtering decomposition (4.10) of* $\omega^{\beta\vartheta}$ *in terms of the orthogonal measure* $\widetilde{\mu}$ *is an extremal decomposition in* $\mathfrak{S}(\mathcal{C})$. *It provides a complete selection of the quasi-particle energies and can be parametrized as*

$$\omega^{\beta\vartheta} = \int_{\mathcal{E}} \omega^\varepsilon d\mu(\varepsilon),\ \omega^\varepsilon \in \partial_e\mathfrak{S}(\mathcal{C}), \tag{4.17}$$

the sector index being $\varrho_0^{\beta\vartheta}$ *for all states involved in the decomposition. This is, why the decomposition corresponds to a decomposition in* $\mathfrak{S}(\mathfrak{A})$.

(ii) *For the GNS-triples it holds the component-wise decomposition*

$$(\Pi_{\beta\vartheta}, \mathcal{H}_{\beta\vartheta}, \Omega_{\beta\vartheta}) = \int_{\mathcal{E}}^{\oplus} (\Pi_\varepsilon, \mathcal{H}_\varepsilon, \Omega_\varepsilon)\,\mathrm{d}\mu(\varepsilon), \tag{4.18}$$

where the Π_ε *are irreducible representations, implying for the associated von Neumann algebras* $\mathfrak{M}_\varepsilon := \Pi_\varepsilon(\mathcal{C})'' = \mathcal{B}(\mathcal{H}_\varepsilon)$.

The GNS-triples in the decomposition may refer to $\mathcal{C}$ *or to* $\mathfrak{A}$.

Now we conclude that each pure energy state ω^ε is invariant as a state on $\mathcal{C}$ under the global dynamical system $(\mathcal{C},\mathbb{R},\tau)$, because its sector index $\varrho_0^{\beta\vartheta}$ and its local density operators (2.22) are time-invariant.

4.3. Definition

In the representation $(\Pi_\varepsilon, \mathcal{H}_\varepsilon, \Omega_\varepsilon)$ *we construct the covariant GNS-representation of the C*-dynamical system* $(\mathcal{C},\mathbb{R},\tau)$ *in the usual way: We define first the unitaries by extension of*

$$U_t^\varepsilon \Pi_\varepsilon(C)\Omega_\varepsilon := \Pi_\varepsilon(\tau_t(C))\Omega_\varepsilon, \quad \forall\, C \in \mathcal{C}, \tag{4.19}$$

and then the selfadjoint generator as the GNS-Hamiltonian of the considered representation, namely

$$K^\varepsilon := \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t} U_t^\varepsilon|_{t=0}, \quad K^\varepsilon\Omega_\varepsilon = 0.$$

The explicit form of the Π_ε-representation dynamics is determined first by the chosen equilibrium sector, with sector index $\varrho_0^{\beta\vartheta}$, and second by the set of indices Λ_-^ε, which indicates the number of γ-occupation operators in the diagonal form of the 'model Hamiltonian' $H_\Lambda^{\beta\vartheta}$ (3.3), surviving multiplication with N_Λ^ε (4.1) from the right. Since in the present Section the ε-index is fixed, we identify here – in contradistinction to the subsequent Section – $\mathfrak{A}$ with $\Pi_\varepsilon(\mathfrak{A})$, writing, e. g., $H_\Lambda^{\beta\vartheta}$ instead of $\Pi_\varepsilon(H_\Lambda^{\beta\vartheta})$.

4.4. Proposition

Let $(\mathfrak{M}_\varepsilon,\mathbb{R},\tau^\varepsilon)$ *be the W*-dynamical system associated with the energy filtered component* ω^ε *of* $\omega^{\beta\vartheta}$, *and let* K^ε *be the implementing GNS-Hamiltonian. Then it holds:*

(i) K^ε *has* $\mathcal{D}_\varepsilon := \mathfrak{A}_0\Omega_\varepsilon$ *as a core and satisfies for* $A \in \mathfrak{A}_\Lambda$

$$\begin{aligned} K^\varepsilon A\Omega_\varepsilon &= [H_\Lambda^{\beta\vartheta}, A]\Omega_\varepsilon = (\hat{H}_\Lambda^{\beta\vartheta} - \sum_{\Lambda_-^\varepsilon} E_\kappa)A\Omega_\varepsilon \\ &= \Big(\sum_{\Lambda_+^\varepsilon} E_\kappa n_\kappa - \sum_{\Lambda_-^\varepsilon} E_\kappa n_\kappa^\perp\Big)A\Omega_\varepsilon, \end{aligned} \tag{4.20}$$

where we have dropped subtraction terms and set

$$\hat{H}_\Lambda^{\beta\vartheta} := H_\Lambda^{\beta\vartheta} + \sum_\Lambda (E_\kappa - \eta_\kappa)/2 = \sum_\Lambda E_\kappa n_\kappa. \quad (4.21)$$

The preceding fact can also be expressed in terms of a strong-resolvent limit of the renormalized model Hamiltonians

$$K^\varepsilon = \text{st-res} - \lim_{\Lambda\to\infty} [\hat{H}_\Lambda^{\beta\vartheta} - \sum_{\Lambda_-^\varepsilon} E_\kappa \mathbb{1}_\varepsilon]. \quad (4.22)$$

(ii) *The set of vectors* $\{\Gamma_\Lambda^{\varepsilon *}\Omega_\varepsilon | \Lambda \in \mathfrak{L}\}$ *in* $\mathcal{H}_\varepsilon$ *(where for* $\Lambda = \varnothing$ *we define* $\Gamma_\Lambda^{\varepsilon *}\Omega_\varepsilon := \Omega_\varepsilon$*) is a complete set of eigenvectors for* K^ε *with the respective eigenvalues* E_Λ^ε *(making the convention* $E_\varnothing^\varepsilon = 0$*).*

(iii) *The representation* $(\Pi_\varepsilon, \mathcal{H}_\varepsilon)$ *is unitarily equivalent to the Fock representation (where* $\varepsilon = \varepsilon^0$*, that is the BCS-groundstate representation), iff* $|\mathcal{K}_-^\varepsilon| < \infty$*.*

(iv) *The W*-dynamical system* $(\mathfrak{M}_\varepsilon, \mathbb{R}, \tau^\varepsilon)$ *satisfies the intersection property of the Borchers-Arveson theorem (cf. Appendix C)*

$$Q_\infty^\varepsilon := \bigcap_{E'\in\mathbb{R}} \Big[\mathfrak{M}^{\tau^\varepsilon}([E';\ \infty))\,\mathcal{H}\Big] = 0,$$

iff $|\mathcal{K}_-^\varepsilon| < \infty$*. If* $|\mathcal{K}_-^\varepsilon| = \infty$ *then* $Q_\infty^\varepsilon = \mathbb{1}$*.*

(v) *If* $|\mathcal{K}_-^\varepsilon| < \infty$ *the Borchers-Arveson Hamiltonian (with zero as lowest spectral value)* H_ε^{BA} *has* $\mathcal{D}_\varepsilon$ *as core and has there the form*

$$\begin{aligned} H_\varepsilon^{BA} A\Omega_\varepsilon &= \text{st-res} - \lim_{\Lambda\to\infty} \hat{H}_\Lambda^{\beta,\vartheta} A\Omega_\varepsilon \\ &= (K^\varepsilon + \sum_{\kappa\in\mathcal{K}_-^\varepsilon} E_\kappa) A\Omega_\varepsilon. \end{aligned} \quad (4.23)$$

Thus, $K^\varepsilon = H_\varepsilon^{BA}$*, iff* $\mathcal{K}_-^\varepsilon = \emptyset$*, that is only in the representation of the BCS-groundstate.*

(vi) *The selfadjoint generator* K^ε *is, however, affiliated with* $\mathfrak{M}_\varepsilon$ *for all* $\varepsilon \in \mathcal{E}$*.*

PROOF:

(a) The Heisenberg dynamics is in the $\varrho_0^{\beta\vartheta}$-sector locally implemented by $H_\Lambda^{\beta\vartheta}$. By derivation one obtains the commutator with $H_\Lambda^{\beta\vartheta}$. Observe that in the GNS-representation over ω^ε it holds in view of (4.1)

$$n_\kappa \Omega_\varepsilon = \begin{cases} \Omega_\varepsilon & \kappa \in \Lambda_-^\varepsilon \\ 0 & \kappa \notin \Lambda_-^\varepsilon. \end{cases} \quad (4.24)$$

Thus we have $H_\Lambda^{\beta\vartheta}\Omega_\varepsilon = \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa \Omega_\varepsilon$, and

$$K^\varepsilon A\Omega_\varepsilon = [H_\Lambda^{\beta\vartheta}, A]\Omega_\varepsilon = (H_\Lambda^{\beta\vartheta} - \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa) A\Omega_\varepsilon.$$

Now, use $n_\kappa = \mathbb{1} - n_\kappa^\perp$ for $\kappa \in \Lambda_-^\varepsilon$ to arrive at the last version of (4.20).

$\mathfrak{A}_0\Omega_\varepsilon$ is a dense linear manifold of vectors, which is invariant under the unitary time translations and thus is a core for their selfadjoint generator.

The strong resolvent limit follows from the fact that on the core $\mathfrak{A}_0\Omega_\varepsilon$ the limit (4.22) exists (in a stationary manner) (cf. [20], Th. VIII.25).

(b) If we apply the elements of the in $\mathfrak{M}_\varepsilon$ total set $\{\Gamma_\Lambda^{\varepsilon *} N_{\Lambda'}^{\varepsilon'} \mid \varepsilon, \varepsilon' \in \mathcal{E},\ \Lambda, \Lambda' \in \mathfrak{L}\}$ to the given cyclic vector Ω_ε, then only the set of (ii) survives.

(c) If $|\mathcal{K}_-^\varepsilon| < \infty$, Ω_ε is an element of the GNS-space over Ω_{ε^0} with ${\varepsilon_\kappa}^0 = \perp, \forall\kappa \in \mathcal{K}$, and is also cyclic, and we are thus in the γ-Fock representation.

If $|\mathcal{K}_-^\varepsilon| = \infty$, the product states ω^ε and ω^{ε^0} are disjoint.

(d) If $|\mathcal{K}_-^\varepsilon| < \infty$, then $K^\varepsilon + \sum_{\mathcal{K}_-^\varepsilon} E_l$ is a positive implementing selfadjoint operator (which gives the cyclic vector Ω_ε the value $\sum_{\mathcal{K}_-^\varepsilon} E_\kappa$). Its spectrum is equal to the Arveson spectrum of the corresponding unitary group, according to Appendix B. By the BA-theorem (cf. Appendix C) we have then $Q_\infty^\varepsilon = 0$.

If $|\mathcal{K}_-^\varepsilon| = \infty$, then we have

$$\bigotimes_{\kappa\in\Lambda_-^\varepsilon} \gamma_k^* [\bigotimes_{\kappa\in\Lambda_-^\varepsilon} \gamma_\kappa \Omega_\varepsilon] = \Omega_\varepsilon, \quad \forall \Lambda_-^\varepsilon \subset \mathcal{K}_-^\varepsilon.$$

The operator $\bigotimes_{\kappa\in K_-^\varepsilon} \gamma_\kappa^*$ is zero as an element of $\mathfrak{M}_\varepsilon$, but the vector in the preceding formula is by construction in $Q_\infty^\varepsilon \mathcal{H}_\varepsilon$ (cf. Appendix C): For each given $E' \in \mathbb{R}$ there is a $\Lambda \in \mathfrak{L}$, such that the algebraic energy of $\bigotimes_{\kappa\in\Lambda_-^\varepsilon} \gamma_\kappa^*$, that is $\sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa(\beta)$ [where $E_\kappa(\beta)$ is the quasi-particle energy from (2.24)], is larger than E'.

If $A \in \mathfrak{A}_0$, then we form the large-energy-operator $A \bigotimes_{\kappa\in\Lambda_-^\varepsilon} \gamma_\kappa^*$ in the above formula and obtain $A\Omega_\varepsilon \in Q_\infty^\varepsilon \mathcal{H}_\varepsilon$. Thus, there is a dense set of vectors in the closed subspace $Q_\infty^\varepsilon \mathcal{H}_\varepsilon$ and $Q_\infty^\varepsilon \mathcal{H}_\varepsilon = \mathcal{H}_\varepsilon$.

(e) According to (i), K^ε generates in any case an inner unitary implementation and is thus affiliated with $\mathfrak{M}_\varepsilon$.

For the physical interpretation one should observe that the individual – that is pure – states of the superconductor are given by the Ω_ε, if one asks for the energy. In this kind of state one has an absolute counting of the quasi particles and their energies. The corresponding Hamiltonian is H^{BA}, whenever it does exist. From this point of view, K^ε is a renormalized Hamiltonian, which also exists in virtue of an infinite renormalization constant, if H^{BA} does not. If $|\mathcal{K}^\varepsilon_-| = \infty$ the assertion $Q^\varepsilon_\infty \mathcal{H}_\varepsilon = \mathcal{H}_\varepsilon$ constitutes a mathematical explication of the physical assertion: All of the representation space $\mathcal{H}_\varepsilon$ has infinite H^{BA}-energy. Algebraic infinite energy elements are not feasible.

5. Extended Temperature Dynamics

We fix β in this and the following Section to a value in (β_c, ∞)!

As we have seen in Section 4, the extremal decomposition of the pure phase states $\omega^{\beta\vartheta}$ into the pure states $\omega^\varepsilon, \varepsilon \in \mathcal{E}$, [cf. (4.17)] is performed by means of the operators in the commutative von Neumann algebra $\widetilde{\mathcal{N}}$, contained in the commutant $\Pi_{\beta\vartheta}(\mathfrak{A})'$.

We define the extension of the equilibrium von Neumann algebra $\mathfrak{M}_{\beta\vartheta} = \Pi_{\beta\vartheta}(\mathfrak{A})'' \subset \mathcal{B}(\mathcal{H}_{\beta\vartheta})$ as follows:

$$\mathfrak{M}^e_{\beta\vartheta} := \widetilde{\mathcal{N}} \vee \mathfrak{M}_{\beta\vartheta} \tag{5.1}$$

(where again the join "$\vee$" denotes the smallest von Neumann algebra containing the two given ones). Since $\mathfrak{M}_{\beta\vartheta}$ is a factor, $\mathfrak{M}^e_{\beta\vartheta}$ has $\widetilde{\mathcal{N}}$ as its (non-trivial) center. The Tomita map: $\Theta_\mu : L^\infty(\mathcal{E}, \mathrm{d}\mu) \overset{onto}{\to} \widetilde{\mathcal{N}}$, constructed in the standard way [11, 18], constitutes a 'diagonalization' of $\widetilde{\mathcal{N}}$ and leads in our context to the unique central decomposition of the extended von Neumann algebra $\mathfrak{M}^e_{\beta\vartheta}$. The latter is in this manner obtained as a direct integral of the type I_∞ factors $\mathfrak{M}_\varepsilon$ over the parameter space $\mathcal{E}$ of possible quasi-particle occupations:

$$\mathfrak{M}^e_{\beta\vartheta} = \int_{\mathcal{E}}^{\oplus} \mathfrak{M}_\varepsilon \, \mathrm{d}\mu(\varepsilon). \tag{5.2}$$

Therefore, $\widetilde{\mathcal{N}}$ performs a system of imprimitivity for the covariant dynamical system [21].

Recalling (2.27), we obtain for the application of the GNS-Hamiltonian $K^{\beta\vartheta}$ onto local excitations [using (3.3)]

$$K^{\beta\vartheta} A \Omega_{\beta\vartheta} = [H^{\beta\vartheta}_\Lambda, A]\Omega_{\beta\vartheta}, \quad A \in \mathfrak{A}_\Lambda.$$

For extending the dynamics to $\mathfrak{M}^e_{\beta\vartheta}$ we need more refined tools from the modular theory of von Neumann algebras, especially the standard form of states in terms of vectors from the natural self-dual cone $\mathcal{P}_{\beta\vartheta} \subset \mathcal{H}_{\beta\vartheta}$. According to [22, 23] and [24] the pointed cone $\mathcal{P}_{\beta\vartheta}$ is the closure of the set $\{Aj(A)\Omega_{\beta\vartheta} \mid A \in \mathfrak{M}_{\beta\vartheta}\}$. For each normal state φ on $\mathfrak{M}_{\beta\vartheta}$ there is a unique $\xi(\varphi) \in \mathcal{P}_{\beta\vartheta}$ with $\langle \varphi ; A \rangle = \langle \xi(\varphi) \mid A\xi(\varphi) \rangle$, $\forall A \in \mathfrak{M}_{\beta\vartheta}$. In consequence to this biunivocal relation there exists also a unique implementation of the temperature Heisenberg dynamics by unitary operators, which leave $\mathcal{P}_{\beta\vartheta}$ invariant. This is just our GNS-implementation $U^{\beta\vartheta}_t$, for which we now conclude (cf. also [11], Section 2.7)

$$\begin{aligned} &U^{\beta\vartheta}_t Aj(B)\Omega_{\beta\vartheta} = \tau^{\beta\vartheta}_t(A) j(\tau^{\beta\vartheta}_t(B))\Omega_{\beta\vartheta}, \\ &A, B \in \mathfrak{M}_{\beta\vartheta},\ \forall t \in \mathbb{R}. \end{aligned} \tag{5.3}$$

Denoting $W^{\beta\vartheta}_{\Lambda t} := \exp[\mathrm{i}tH^{\beta\vartheta}_\Lambda] \in \mathfrak{A}_\Lambda \subset \mathfrak{M}_{\beta\vartheta}$, we derive

$$\begin{aligned} &U^{\beta\vartheta}_t Aj(B)\Omega_{\beta\vartheta} = \\ &W^{\beta\vartheta}_{\Lambda t} j(W^{\beta\vartheta}_{\Lambda t})(Aj(B)) W^{\beta\vartheta *}_{\Lambda t} j(W^{\beta\vartheta *}_{\Lambda t})\Omega_{\beta\vartheta}, \\ &A, B \in \mathfrak{A}_\Lambda. \end{aligned} \tag{5.4}$$

The above mentioned uniqueness of the standard representation of normal states by vectors in $\mathcal{P}_{\beta\vartheta}$ leads to

$$W^{\beta\vartheta}_{\Lambda t} j(W^{\beta\vartheta}_{\Lambda t})\Omega_{\beta\vartheta} = \Omega_{\beta\vartheta}, \forall t \in \mathbb{R}, \forall \Lambda \in \mathfrak{L}. \tag{5.5}$$

Differentiation of (5.4) to t gives

$$\begin{aligned} &K^{\beta\vartheta} Aj(B)\Omega_{\beta\vartheta} = [H^{\beta\vartheta}_\Lambda - j(H^{\beta\vartheta}_\Lambda)] Aj(B)\Omega_{\beta\vartheta}, \\ &A, B \in \mathfrak{A}_\Lambda. \end{aligned}$$

Since $W^{\beta\vartheta}_{\Lambda t} j(W^{\beta\vartheta}_{\Lambda t}) \in \mathfrak{M}^e_{\beta\vartheta}$, $\forall \Lambda \in \mathfrak{L}$, their net-limit $U^{\beta\vartheta}_t$, in the strong operator topology for increasing Λ, is also in $\mathfrak{M}^e_{\beta\vartheta}$. Thus

$$\begin{aligned} K^{\beta\vartheta} &= \text{st-res} - \lim_\Lambda \, [H^{\beta\vartheta}_\Lambda - j(H^{\beta\vartheta}_\Lambda)] \\ &= \text{st-res} - \lim_\Lambda \sum_{\kappa \in \Lambda} E_\kappa(\beta)[n_\kappa - j(n_\kappa)]. \end{aligned} \tag{5.6}$$

In terms of the foregoing formula, the GNS-Hamiltonian of the temperature representation, that is the selfadjoint operator $K^{\beta\vartheta}$, is displayed as the strong-resolvent-limit of the local model Hamiltonians (3.3) minus *operator renormalization terms*.

We are thus confronted with a generalization of the renormalization concept in irreducible representations, where the diverging nets of c-numbers are now replaced by diverging nets of operators from the center (resp. from the commutant). The latter display a whole spectrum of c-number counter terms, which we are going to analyze.

Mathematically the foregoing formula exhibits that $K^{\beta\vartheta}$ is affiliated with $\mathfrak{M}^e_{\beta\vartheta}$. One sees that the extended von Neumann algebra $\mathfrak{M}^e_{\beta\vartheta}$ may also be generated by $\mathfrak{M}_{\beta\vartheta}$ and the spectral projections of $K^{\beta\vartheta}$ resp. of the $U^{\beta\vartheta}_t$. We may relate our extended dynamical system with the theory of crossed products (cf., e. g., [21, 25, 26]).

5.1. Observation

The W-dynamical system* $(\mathfrak{M}^e_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta e})$ *comes from the C*-crossed product, induced by the action of* $\mathbb{R}$ *in terms of* $*$*-automorphisms in* $\Pi_{\beta\vartheta}(\mathfrak{A})$*, by performing the w*-closure. The extended von Neumann algebra* $\mathfrak{M}^e_{\beta\vartheta}$ *is, therefore, the crossed product of the algebra* $\mathfrak{M}_{\beta\vartheta}$ *by the abelian group* $\mathbb{R}$ *with respect to* $\tau^{\beta\vartheta}$ *(note that* $\mathbb{R} \cong \hat{\mathbb{R}}$*):*

$$\mathfrak{M}^e_{\beta\vartheta} = \mathfrak{M}_{\beta\vartheta} \vee U^{\beta\vartheta}_{\mathbb{R}} = \mathfrak{M}_{\beta\vartheta} \times_{\tau^{\beta\vartheta}} \mathbb{R}.$$

Now we may analyze the dynamics on the extended algebra $\mathfrak{M}^e_{\beta\vartheta}$, which is naturally introduced by setting

$$\tau^{\beta\vartheta e}_t(A) := U^{\beta\vartheta}_t A U^{\beta\vartheta *}_t, \quad \forall\, A \in \mathfrak{M}^e_{\beta\vartheta}. \tag{5.7}$$

We check that $\Omega_{\beta\vartheta}$ is separating (and cyclic) for $\mathfrak{M}^e_{\beta\vartheta}$ and that $\tau^{\beta\vartheta e}$ is the modular automorphism group for $(\mathfrak{M}^e_{\beta\vartheta}, \Omega_{\beta\vartheta})$. Part of the following assertions may be, however, directly verified:

5.2. Proposition

The dynamics $\tau^{\beta\vartheta e}_t \in \mathrm{Aut}(\mathfrak{M}^e_{\beta\vartheta})$ *leaves the center* $\tilde{\mathcal{N}} = \mathfrak{M}^e_{\beta\vartheta} \cap \mathfrak{M}^{e\prime}_{\beta\vartheta}$ *pointwise invariant and decomposes along the central decomposition of* $\mathfrak{M}^e_{\beta\vartheta}$*:*

$$\begin{aligned} \tau^{\beta\vartheta e}_t(A) &= \int_{\mathcal{E}}^{\oplus} \tau^{\varepsilon}_t(A_\varepsilon) \mathrm{d}\mu(\varepsilon), \\ \forall A &= \int_{\mathcal{E}}^{\oplus} A_\varepsilon \mathrm{d}\mu(\varepsilon) \in \mathfrak{M}^e_{\beta\vartheta}, \end{aligned} \tag{5.8}$$

with the previous $\tau^{\varepsilon}_t \in \mathrm{Aut}(\mathfrak{M}_\varepsilon)$.

PROOF: Argue similar to [27].

Let $\mathcal{E}_0 \subset \mathcal{E}$ be a measurable subset with $\mu(\mathcal{E}_0) > 0$ and define:

$$\Gamma^*_\Lambda(\mathcal{E}_0) := \int_{\mathcal{E}_0}^{\oplus} \Gamma^{\varepsilon *}_\Lambda \,\mathrm{d}\mu(\varepsilon), \tag{5.9}$$

$$\Gamma^*_\Lambda := \Gamma^*_\Lambda(\mathcal{E}) := \int_{\mathcal{E}}^{\oplus} \Gamma^{\varepsilon *}_\Lambda \,\mathrm{d}\mu(\varepsilon), \tag{5.10}$$

$$P(\mathcal{E}_0) := \int_{\mathcal{E}_0}^{\oplus} \mathbb{1}_\varepsilon \,\mathrm{d}\mu(\varepsilon). \tag{5.11}$$

Now remark that

$$\Gamma^*_\Lambda(\mathcal{E}_0) = P(\mathcal{E}_0)\Gamma^*_\Lambda \tag{5.12}$$

due to the pointwise multiplication under the direct integral.

5.3. Proposition

(i) *An element* $A \in \mathfrak{M}^e_{\beta\vartheta}$ *is in the spectral subspace* $\mathfrak{M}^e_{\beta\vartheta}(B)$*,* B *a closed set in* $\hat{\mathbb{R}}$*, iff*

$$A = \int_{\mathcal{E}_0}^{\oplus} A_\varepsilon \mathrm{d}\mu(\varepsilon), \quad A_\varepsilon \in \mathfrak{M}_\varepsilon(B), \tag{5.13}$$

for a set $\mathcal{E}_0$ *with finite* μ*-measure.*

(ii) *The following relation holds:*

$$\overline{\mathfrak{M}_{\beta\vartheta}(B)\mathcal{H}_{\beta\vartheta}}^{\sigma} = \overline{\mathfrak{M}^e_{\beta\vartheta}(B)\mathcal{H}_{\beta\vartheta}}^{\sigma}. \tag{5.14}$$

(iii) *It holds*

$$\begin{aligned} Q^{\beta\vartheta}_\infty &:= \bigcap_{E\in\mathbb{R}} \left[\mathfrak{M}^{\beta\vartheta}([E;\ \infty))\ \mathcal{H}_{\beta\vartheta}\right] \\ &= \bigcap_{E\in\mathbb{R}} \left[\mathfrak{M}^{\beta\vartheta e}([E;\infty))\mathcal{H}_{\beta\vartheta}\right] =: Q^{\beta\vartheta e}_\infty. \end{aligned} \tag{5.15}$$

PROOF:

(i) The condition (5.13) is clearly sufficient for $A \in \mathfrak{M}^e_{\beta\vartheta}(B)$. On the other hand, if there is an $\mathcal{E}_1 \subset \mathcal{E}$ with finite μ-measure and such that $Sp(A_\varepsilon)$ is not contained in B, $\forall \varepsilon \in \mathcal{E}_1$, then we have an $f \in L^1(\mathbb{R})$ with $supp \hat{f} \cap B = \varnothing$ and $\tau_f(A) \neq 0$, which contradicts the criterion of Lemma A.3.

(ii) Show that the set of linear combinations L of elements from $\mathfrak{M}_{\beta\vartheta}(B)\cdot\mathcal{N}$ is strongly dense in $\mathfrak{M}^e_{\beta\vartheta}(B)$. Then any vector $\Psi \in \left[\mathfrak{M}^e_{\beta\vartheta}(B)\ H_{\beta\vartheta}\right]$ is the limit of a sequence $\{L_n\Psi_n\} \subset \mathfrak{M}_{\beta\vartheta}(B)\ H_{\beta\vartheta}$.

(iii) This is directly implied by the foregoing result.

We shall make heavy use of the following conclusion:

5.4. Corollary

Let $E^{\varepsilon}_{\Lambda} \geqslant E$ for all $\varepsilon \in \mathcal{E}_0$ with $\mu(\mathcal{E}_0) > 0$, then it holds $\Gamma^*_{\Lambda}(\mathcal{E}_0) \in \mathfrak{M}^e_{\beta\vartheta}([E,\infty))$.

6. Energy Renormalization for Finite Temperatures

For studying the renormalized Hamiltonians in the finite temperature case we use the extended W*-dynamical system $(\mathfrak{M}^e_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta e})$. It is well-known from general KMS- resp. Tomita-Takesaki-theory that the dynamics in the temperature representation has an implementing selfadjoint generator with two-sided unbounded spectrum. But this implementation is not the only one, and the Borchers-Arveson theorem does not, in principle, exclude another unitary implementation with positive spectrum, or positive spectrum in a reducing subspace, as long as one does not know whether the intersection projection $Q^{\beta\vartheta}_{\infty}$ in the GNS-Hilbert space $\mathcal{H}_{\beta\vartheta}$ (cf. Appendix C) equals unity, or not. We study in this Section the occupation probabilities for the quasi-particle excitations in order to obtain the distribution of quasi-particle energy values and to gain information on $Q^{\beta\vartheta}_{\infty}$.

6.1. Definition (Asymptotic Occupation Sets)

(i) We start with a sequence $\{\Lambda_n \mid n \in \mathbb{N}\} \subset \mathfrak{L}$ with the following properties

(a) $|\Lambda_n| = n, \qquad \forall\, n \in \mathbb{N}$;

(b) the Λ_n are pair-wise disjoint;

(c) for each $\Lambda_0 \in \mathfrak{L}$ there is an $n_0 \in \mathbb{N}$ with $\Lambda_0 \cap \Lambda_n = \emptyset, \forall n \geq n_0$.

(ii) With this and for a given $\alpha \in (0,1)$ we define a sequence of measurable subsets $\mathcal{E}^{\alpha}_n$ of $\mathcal{E}$ by

$$\mathcal{E}^{\alpha}_n := \{\varepsilon \in \mathcal{E} \,|\, |\Lambda^{\varepsilon}_{n-}| = [\alpha n]\}, \forall\, n \in \mathbb{N},$$

where $[\alpha n]$ denotes the least upper integer next to $\alpha n, \ n \in \mathbb{N}$. Since these sets are pair-wise μ-independent [μ from (4.13)] they may serve to construct terminal events (cf., e. g., [28]).

(iii) We define

$$\mathcal{E}^{\alpha} := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{E}^{\alpha}_n,$$

which is the subset of $\mathcal{E}$ of those ε, which fulfill finally the relation $|\Lambda^{\varepsilon}_{n-}| = [\alpha n]$, for increasing n.

In the following Lemma mathematical techniques come into play which are typical for proving 0-1-laws.

6.2. Lemma

For a BCS-model of the considered class, with μ the extremal measure from (4.13) on the parameter space $\mathcal{E}$, decomposing the temperature pure phase state, it holds:

(i)

$$\begin{aligned} \lim_{\kappa\to\infty} g_{\kappa} &=: g = g(\beta) \\ &= \exp(-\beta E_0)/[1+\exp(-\beta E_0)], \end{aligned} \tag{6.1}$$

$$\begin{aligned} \lim_{\kappa\to\infty} g^{\perp}_{\kappa} &=: g^{\perp} = 1 - g(\beta) \\ &= 1/[1+\exp(-\beta E_0)], \end{aligned} \tag{6.2}$$

where on the r. h. s. are the Fermi distribution functions for one quasi-particle of the homogenized model.

(ii)

$$\lim_{n\to\infty} \mu(\mathcal{E}^{\alpha}_n) = \mu(\mathcal{E}^{\alpha}) = \begin{cases} 1, & \text{if} \quad \alpha = g \\ 0, & \text{if} \quad \alpha \neq g \end{cases}.$$

(iii)

$$\begin{aligned} st-\lim_{n\to\infty} P(\mathcal{E}^{\alpha}_n) &:= st-\lim_{n\to\infty} \Gamma^*_{\Lambda_n}(\mathcal{E}^{\alpha}_n)\Gamma_{\Lambda_n}(\mathcal{E}^{\alpha}_n) \\ &= P(\mathcal{E}^{\alpha}) = \begin{cases} 1, & \text{if} \quad \alpha = g \\ 0, & \text{if} \quad \alpha \neq g \end{cases}. \end{aligned}$$

PROOF:

(i) From our model assumptions the model parameters, the gaps and thus the quasi-particle energies and g_{κ} tend to their resp. homogeneous values for $\kappa \to \infty$.

(ii) Consider $\mathcal{E}^{\alpha}_n$ as a finite union of cylinder sets

$$\mathcal{Z}_{\varepsilon}(\Lambda_n) := \{\varepsilon' \in \mathcal{E} \mid \ r_{\Lambda_n}(\varepsilon') = r_{\Lambda_n}(\varepsilon)\}, \tag{6.3}$$

namely,

$$\mathcal{E}^{\alpha}_n = \dot{\bigcup_{\varepsilon\in\mathcal{E},\ |\Lambda^{\varepsilon}_{n-}|=[\alpha n]}} \mathcal{Z}_{\varepsilon}(\Lambda_n).$$

Since the measure μ has the product property, we find with (4.2)

$$\begin{aligned} \mu(\mathcal{Z}_{\varepsilon}(\Lambda_n)) &= \prod_{\kappa\in\Lambda_n} g^{\varepsilon_{\kappa}}_{\kappa} = (\Omega_{\beta\vartheta} \mid N^{\varepsilon}_{\Lambda_n}\, \Omega_{\beta\vartheta}), \\ \mu(\mathcal{E}^{\alpha}_n) &= \sum_{\varepsilon\in\mathcal{E},\ |\Lambda^{\varepsilon}_{n-}|=[\alpha n]} \ \prod_{\kappa\in\Lambda_n} g^{\varepsilon_{\kappa}}_{\kappa}. \end{aligned}$$

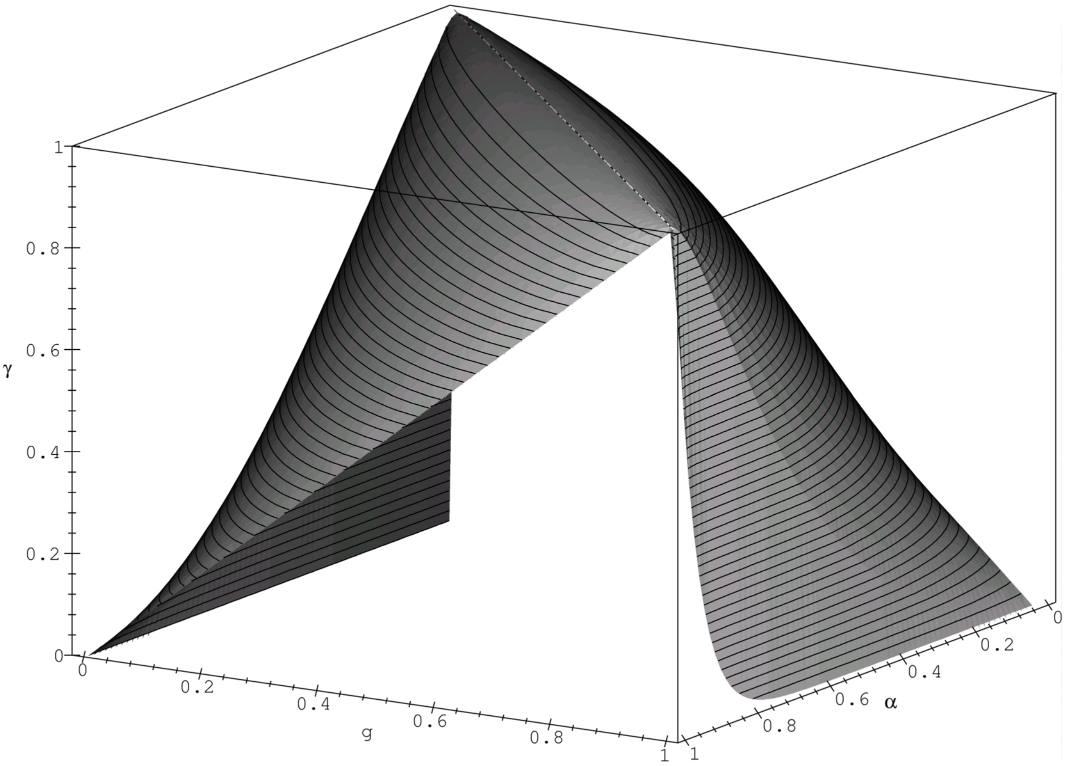

Fig. 1. The function $\gamma = \gamma(g, \alpha)$ to determine $\mu(\mathcal{E}_n^\alpha)$.

Since

$$\mathrm{card}\{\varepsilon \in \mathcal{E} \mid |\Lambda_{n-}^\varepsilon| = [\alpha n]\} = \binom{n}{[\alpha n]},$$

we have for large n (implying large $[\alpha n]$) in virtue of Stirling's formula

$$\begin{aligned}\mu(\mathcal{E}_n^\alpha) &\approx \binom{n}{[\alpha n]} g^{n\alpha}(1-g)^{(1-\alpha)n} \\ &= \frac{n! g^{n\alpha}(1-g)^{(1-\alpha)n}}{(\alpha n)!(n-\alpha n)!} \\ &\approx \frac{n^n \mathrm{e}^{-n} g^{n\alpha}(1-g)^{(1-\alpha)n}}{(\alpha n)^{\alpha n}\mathrm{e}^{-\alpha n}[(1-\alpha)n]^{(1-\alpha)n}\mathrm{e}^{-(1-\alpha)n}} \\ &= \frac{g^{n\alpha}(1-g)^{(1-\alpha)n}}{\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}} := \gamma(g;\alpha)^n.\end{aligned}$$

The dropping of the bracket in $[\alpha n]$ can be justified by a more careful handling of the asymptotics for large n. As illustrated in Fig. 1, the function $\gamma(g;\alpha)^n$ is strictly smaller than unity for $\alpha \neq g$ and equal to unity for $\alpha = g$. What we make use of, is the arbitrary nearness of $\mu(\mathcal{E}_n^\alpha)$ to $\gamma(g;\alpha)^n$. Thus the sum $\sum_n \mu(\mathcal{E}_n^\alpha)$ approaches for large n the geometric series over $\gamma(g;\alpha)^n$. Observing the independence of the $\mathcal{E}_n^\alpha$, the lemma of Borel-Cantelli, [28] Ch. II, tells us then, that in the first case with converging series it follows $\mu(\mathcal{E}^\alpha) = 0$, whereas in the second case with diverging series it follows $\mu(\mathcal{E}^\alpha) = 1$. Now, we have in virtue of the lemma of Fatou [29] and of the finiteness of the measure μ

$$\mu(\mathcal{E}^\alpha) \le \limsup_{n\to\infty} \mu(\mathcal{E}_n^\alpha).$$

In the first case, the series on the r.h.s. is monotonously decreasing to zero, whereas in the second case $\mu(\mathcal{E}_n^\alpha)$ has still only one accumulation point in virtue of our model assumptions. Thus this must be $\limsup_{n\to\infty}\mu(\mathcal{E}_n^\alpha)$, and the sequence of measures has to converge to unity.

(iii) We have by definition

$$P(\mathcal{E}_n^\alpha) = \int_{\mathcal{E}_n^\alpha}^\oplus \mathbb{1}_\varepsilon d\mu(\varepsilon).$$

For $A, B \in \mathfrak{A}_\Lambda$ and n large enough we find

$$\begin{aligned}\langle AP(\mathcal{E}_n^\alpha)\Omega \mid B\Omega\rangle &= \langle A\Omega \mid B\Omega\rangle\,\langle\Omega \mid P(\mathcal{E}_n^\alpha)\Omega\rangle \\ &= \langle A\Omega \mid B\Omega\rangle\,\mu(\mathcal{E}_n^\alpha) \\ &\longrightarrow \begin{cases}\langle A\Omega \mid B\Omega\rangle & \text{if } \alpha = g \\ 0 & \text{if } \alpha \neq g\end{cases}.\end{aligned}$$

Thus the projections converge in the weak and, by squaring, in the strong Hilbert space topology.

6.3. Remark

In the preceding Lemma the support of the product measure μ on the σ-algebra $\Sigma(\mathcal{E})$ has been characterized by its asymptotic nearness to the product measure μ_0 of the homogenized BCS-model. The homogeneous product measure determines the type of the thermal von Neumann algebra, not only for the homogeneous but also for each inhomogeneous model, for which it is the homogeneous average. But it is not true that each inhomogeneous measure μ is equivalent to the coresponding homogeneous measure. The fluctuations of the quasi-particle energies may be so strong (for an allowed model), that the inhomogeneous and the homogeneous measures become disjoint. (For this one may apply the Kakutani Theorem for the comparison of product measures, given in [30].)

The renormalized GNS-Hamiltonian for the system writes according to (5.6), inserting the central decomposition of the operators,

$$K^{\beta\vartheta} = \text{st-res} - \lim_\Lambda \sum_{\kappa\in\Lambda} E_\kappa(\beta)[n_\kappa - j(n_\kappa)] \quad (6.4)$$

$$= \text{st-res} - \lim_\Lambda \sum_{\kappa\in\Lambda} \int_{\mathcal{E}}^{\oplus} E_\kappa(\beta)[\Pi_\varepsilon(n_\kappa) - \nu_\kappa(\varepsilon)\mathbb{1}_\varepsilon]\,\mathrm{d}\mu(\varepsilon), \quad (6.5)$$

where we have used for the inverse Tomita image (4.16) of the central element $j(n_\kappa)$ the notation

$$[\Theta_\mu^{-1}(j(n_\kappa))](\varepsilon) =: \nu_\kappa(\varepsilon) = \begin{Bmatrix} 1, & \kappa\in\mathcal{K}_-^\varepsilon, \\ 0, & \kappa\in\mathcal{K}_+^\varepsilon \end{Bmatrix}. \quad (6.6)$$

6.4. Theorem

Consider a BCS-model of the allowed class in the finite temperature representation. Determine the decomposition measure μ of the pure phase state and the supporting set $\mathcal{E}^g$ from Lemma 6.2.

(i) *For the GNS-Hamiltonian holds the local approximation and the direct integral decomposition*

$$\begin{aligned}K^{\beta\vartheta} &= \text{st-res} \\ &-\lim_\Lambda \int_{\mathcal{E}^g}^{\oplus} \Big[\sum_{\kappa\in\Lambda} E_\kappa \Pi_\varepsilon(n_\kappa) - \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa \mathbb{1}_\varepsilon\Big]\,\mathrm{d}\mu(\varepsilon) \\ &= \int_{\mathcal{E}^g}^{\oplus} K^\varepsilon\,\mathrm{d}\mu(\varepsilon).\end{aligned} \quad (6.7)$$

This makes explicit, that only infinite energy renormalization terms occur within the operator subtraction terms, since for all ε in the supporting set $\mathcal{E}^g$ the limiting set $\mathcal{K}_-^\varepsilon$ for the Λ_-^ε is infinite.

(ii) *Each $0 \neq \Psi \in \mathcal{H}_{\beta\vartheta}$ contains both infinitely many quasi-particles and infinitely many quasi-holes, the ratio of the total quasi-particle number resp. of the total hole number against the total particle number being $g(\beta)$ resp. $1 - g(\beta)$. (States which do not satisfy these conditions have zero norm.)*

(iii) *The BA-intersection space has the form*

$$\bigcap_{E\in\mathbb{R}} \overline{\mathfrak{M}_{\beta\vartheta}^{(e)}([E,\infty))\,\mathcal{H}_{\beta\vartheta}} = Q_\infty^{\beta\vartheta(e)}\mathcal{H}_{\beta\vartheta} = \mathcal{H}_{\beta\vartheta}. \quad (6.8)$$

There is no non-trivial subspace of $\mathcal{H}_{\beta\vartheta}$, in which a unitary implementation for the finite temperature dynamics is possible with its selfadjoint generator displaying a lower bounded spectrum.

PROOF:

(a) The first equation follows from (6.4) and the observation, that $\nu_\kappa(\varepsilon) = 1$ under the subsidiary condition $\kappa \in \Lambda$, iff $\kappa \in \Lambda_-^\varepsilon$ and from the fact that $\mu(\mathcal{E}^g) = 1$. The second equation arises from the interchange of the st-res-limit with the integration, the latter being justified by the Lebesgue dominated convergence theorem.

(b) We observe that for $\alpha = g$ it holds $P(\mathcal{E}_n^\alpha)\mathcal{H}_{\beta\vartheta} \subset \mathcal{H}_n := \overline{\mathfrak{M}_{\beta\vartheta}^e(([\alpha n],\infty))\mathcal{H}_{\beta\vartheta}}$ and that the $\mathcal{H}_n$ are monotonously decreasing for increasing n. For a given $\Psi \in \mathcal{H}_{\beta\vartheta}$ there cannot be a finite distance δ to an $\mathcal{H}_n$, since – in virtue of Lemma 6.2 (iii) – there is always an $m \in \mathbb{N}$ such that $\|\Psi - P(\mathcal{E}_m^\alpha)\Psi\| <$

δ, $P(\mathcal{E}_m^\alpha)\Psi$ being in $\mathcal{H}_n$. Thus, $\Psi \in \mathcal{H}_n,\ \forall n \in \mathbb{N}$. Thus $\Psi \in \cap_n \mathcal{H}_n,\ \forall \Psi \in \mathcal{H}_{\beta\vartheta}$. From this follows (6.8), since for each $E \in \mathbb{R}$, $\mathfrak{M}^e_{\beta\vartheta}([E,\infty)) \subset \mathfrak{M}^e_{\beta\vartheta}([E_n,\infty))$ for some $n \in \mathbb{N}$. This consideration, combined with the result that $P(\mathcal{E}_n^\alpha) \to 0$, for $\alpha \neq g(\beta)$, also justifies the range of integration in (i), since it states that each $0 \neq \Psi \in \mathcal{H}_{\beta\vartheta}$ eventually is in $P(\mathcal{E}_n^\alpha)\mathcal{H}_{\beta\vartheta}$ if and only if $\alpha = g(\beta)$.

(c) With the preceding result assertion (iii) follows directly from the BA-theorem for the total dynamical system. If Q is an invariant projection, then its central decomposition may be restricted to $\mathcal{E}^g$ and the by Q reduced dynamical system does not have the BA-intersection property either.

For a comparison of the operator algebraic formulation of a BCS-model with its usual treatment in theoretical physics, where a 'large' but finite number of particles is taken into account, let us restrict the dynamical system $(\mathfrak{M}^e_{\beta\vartheta}, \mathbb{R}, \tau^{\beta\vartheta e})$ to a finite set Λ of quasi-particles (after having first performed the thermodynamical limit). Specifying the quasi-particle occupation $r_\Lambda(\varepsilon)$ in Λ leads to the cylinder set $\mathcal{Z}_\varepsilon(\Lambda) \subset \mathcal{E}$ from (6.3) with dynamically invariant projection $P_\varepsilon(\Lambda) = \int^\oplus_{\mathcal{Z}_\varepsilon(\Lambda)} \mathbb{1}_\varepsilon \, d\mu(\varepsilon)$. Thus, the total system is already infinite, but we look only at a finite part of it and discover there with the finite probability $\Pi_\Lambda g_\kappa^{\varepsilon_\kappa}$ the occupation configuration $r_\Lambda(\varepsilon)$. Mathematically, the Arveson spectral theory may easily be taken over to the reduced W*-dynamical system $(P_\varepsilon(\Lambda)\mathfrak{M}^e_{\beta\vartheta}P_\varepsilon(\Lambda), \mathbb{R}, P_\varepsilon(\Lambda)\tau^{\beta\vartheta e}P_\varepsilon(\Lambda))$. The spectral values depend on β, since we are dealing with an infinite system, whose subsystems are related by the thermally averaged meanfield interaction. Using this kind of projection leaves the local observables of the particles outside of Λ, nevertheless, completely unspecified.

Prescribing the thermal average for the observables in the complementary index domain leads to the conditional expectation

$$P_\varepsilon^{\beta\vartheta}(\Lambda) : \mathfrak{M}^e_{\beta\vartheta} \to \Pi_\varepsilon(\mathfrak{A}_\Lambda) \tag{6.9}$$

$$P_\varepsilon^{\beta\vartheta}(\Lambda)(A) = \int^\oplus_{\mathcal{Z}_\varepsilon(\Lambda)} \left\langle \Omega_\varepsilon^{\Lambda^c} \mid A_\varepsilon\, \Omega_\varepsilon^{\Lambda^c} \right\rangle \mathrm{d}\mu(\varepsilon),$$

$$A = \int^\oplus_{\mathcal{E}} A_\varepsilon \, \mathrm{d}\mu(\varepsilon) \ \in \mathfrak{M}^e_{\beta\vartheta}, \tag{6.10}$$

where $\Omega_\varepsilon^{\Lambda^c}$ denotes the product vector Ω_ε, restricted to the index domain Λ^c. To illustrate our notation consider the application of $P_\varepsilon^{\beta\vartheta}(\Lambda)$ onto the elements $n_\kappa = \int^\oplus_{\mathcal{E}} \Pi_\varepsilon(n_\kappa)\, \mathrm{d}\mu(\varepsilon)$ resp. $j(n_\kappa) = \int^\oplus_{\mathcal{E}} \nu_\kappa(\varepsilon)\mathbb{1}_\varepsilon \, \mathrm{d}\mu(\varepsilon)$, leading to

$$\begin{aligned} &P_\varepsilon^{\beta\vartheta}(\Lambda)(n_\kappa \text{ resp.} j(n_\kappa)) \\ &\quad = \begin{Bmatrix} \Pi_\varepsilon(n_\kappa) \text{ resp.}\, \nu_\kappa(\varepsilon)\mathbb{1}_\varepsilon & \mid \kappa \in \Lambda \\ \bar{n}_\kappa^{\beta\vartheta}\, \mathbb{1}_\varepsilon & \mid \kappa \notin \Lambda \end{Bmatrix}, \end{aligned}$$

where the bar denotes the thermal average. By extending its domain we define the application of the conditional expectation to the GNS-Hamiltonian

$$\begin{aligned} K_\Lambda^{\beta\vartheta\varepsilon} &:= \text{st-res} - \lim_{\Lambda'} P_\varepsilon^{\beta\vartheta}(\Lambda)(K_{\Lambda'}^{\beta\vartheta}) \\ &= \sum_{\kappa\in\Lambda} E_\kappa \Pi_\varepsilon(n_\kappa) - \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa \mathbb{1}_\varepsilon. \end{aligned} \tag{6.11}$$

The subtraction term takes account for the holes, indexed by $\kappa \in \Lambda_-^\varepsilon$. The Borchers-Arveson Hamiltonian exists (only) locally and is obtained by shifting the energy scale by the total hole energy:

$$H_\Lambda^{BA\varepsilon} = K_\Lambda^{\beta\vartheta\varepsilon} + \sum_{\kappa\in\Lambda_-^\varepsilon} E_\kappa\, \mathbb{1}_\varepsilon.$$

There is no opportunity to bring into play the pure c-number subtraction term $\sum_{\kappa\in\Lambda} (E_\kappa - \eta_\kappa)/2\, \mathbb{1}_\varepsilon$ of the model Hamiltonian (3.3). The latter corresponds to the so-called condensation energy of [6] (the slight deviation being due to a different mean field ansatz in [6], and the factor $1/2$ stems from our single particle, instead of pair, counting). The decisive point is, that the condensation energy, diverging with increasing Λ, cannot be incorporated into any renormalized Hamiltonian, since it does not fit into any sharp energy representation, which lies within the support of the thermal measure.

6.5. Conclusion

The GNS-Hamiltonian $K^{\beta\vartheta}$ in the thermal representation is the statistical superposition of BA-Hamiltonians, which count the energy above the vacuum, with diverging subtraction constants.

The family of energy subtraction terms make up the diverging operator subtraction, which is canonically obtained by the standard implementation of automorphisms, and its generators, in the thermal von Neumann algebra $\mathfrak{M}_{\beta\vartheta}$ and which prevents $K^{\beta\vartheta}$ to be affiliated with $\mathfrak{M}_{\beta\vartheta}$. ($K^{\beta\vartheta}$ is affiliated with $\mathfrak{M}^e_{\beta\vartheta}$.)

Since the statistical superposition of physically interpretable energy operators does not change the meaning of the quantity, the thermal Hamiltonian $K^{\beta\vartheta}$ *represents a physically interpretable energy observable.*

For a further illustration of the occupation statistics let us formulate the following:

6.6. Observation (Quasi Pair Occupation)

In virtue of our numbering, described at the beginning of Section 3, the occupation probability of a quasi pair with time-reversed quantum numbers for the constituting quasi-particle components has the form (assuming in the following that κ *counts always a 0-quasi-particle)*

$$\left\langle \omega^{\beta\vartheta} ; n_\kappa n_{\kappa+1} \right\rangle = g_\kappa^2 \to g^2 > 0, \text{for } \kappa \to \infty.$$

Therefore $\sum_\kappa \mu(\mathcal{E}_\kappa^p :=$ *one pair at* κ *is occupied*$) = \infty$ *and from the Borel-Cantelli lemma we conclude via the pair-wise independence of the* $\mathcal{E}_\kappa^p$

$$\mu(\text{infinitely many quasi pairs are occupied}) = \mu\Big(\bigcap_{\lambda=1}^{\infty} \bigcup_{\kappa=\lambda}^{\infty} \mathcal{E}_\kappa^p\Big) = 1.$$

That infinite occupation of quasi-particle pairs with probility 1 does, however, not signify a condensation of quasi pairs follows from the fact $\left\langle \omega^{\beta\vartheta} ; \gamma_\kappa \gamma_{\kappa+1} \right\rangle = 0, \forall\ \kappa\ \in \mathcal{K}$. *As is well known, the nonvanishing of the corresponding expectation for electron pairs makes up the order parameter of a superconductor and represents phase ordering. The different values of the two pair expectations signifies the discontinuity of the Bogoliubov transformation in the* σ*-weak topology in contradistinction to its norm-continuity.*

Appendix

A. The Arveson Spectral Theory – Basic Definitions

For the convenience of the reader, let us compile some basic concepts of the Arveson spectral theory [12]. This theory was elaborated for locally compact groups, which act via isometries in general Banach spaces in a continuous manner with respect to certain weak topologies. We will specialize in this subsection to the group $\mathbb{R}$ (the real time axis) – with the isomorphic dual group $\hat{\mathbb{R}}$ (the real energy axis) – acting $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-continuously via *-automorphisms in a W*-algebra $\mathfrak{M}$ with the pre-dual space $\mathfrak{M}_*$. The general group theoretic duality relation takes the form

$$\langle E ; t \rangle = \exp(\mathrm{i} t E), \quad \forall E \in \hat{\mathbb{R}} \text{ and } \forall t \in \mathbb{R}.$$

The Arveson spectral theory is connected with the left regular representation of $\mathbb{R}$ on the group algebra $L^1(\mathbb{R})$ (with convolution as multiplication), where the latter functions serve to "smear" the time dependence of the $*$-automorphisms. More precisely, for every $f \in L^1(\mathbb{R})$ there is a map

$$\tau_f : \mathfrak{M} \longrightarrow \mathfrak{M},\ \tau_f(A) := \int_{\mathbb{R}} \tau_t(A) f(t)\, \mathrm{d}t,\ A \in \mathfrak{M}, \tag{A.1}$$

where the integral exists in the σ-weak topology. This gives rise to the norm decreasing homomorphism $\overline{\tau}$ from $L^1(\mathbb{R})$ into the σ-weakly continuous operators on $\mathfrak{M}$

$$\overline{\tau} : L^1(\mathbb{R}) \longrightarrow \mathfrak{B}_\sigma(\mathfrak{M}),\ \ \overline{\tau}(f) := \tau_f. \tag{A.2}$$

It holds

$$\|\tau_f(A)\| \le \|f\|_1 \|A\|, \qquad \forall f \in L^1(\mathbb{R}),\ \forall A \in \mathfrak{M}.$$

A.1. Definition

Let be $t \mapsto \tau_t$ *a* $\sigma(\mathfrak{M}, \mathfrak{M}_*)$*-continous representation of* $\mathbb{R}$ *in terms of automorphisms of the W*-algebra* $\mathfrak{M}$*, and* $\mathfrak{Y}$ *an arbitrary subset of* $\mathfrak{M}$*; then we define:*

(i) *a closed* $*$*-ideal* $\mathfrak{I}_{\mathfrak{Y}}^\tau$ *of* $L^1(\mathbb{R})$ *by*

$$\mathfrak{I}_{\mathfrak{Y}}^\tau := \{ f \in L^1(\mathbb{R}) \mid \tau_f(A) = 0,\ \forall A \in \mathfrak{Y} \};$$

(ii) *the (local) Arveson spectrum of* $\mathfrak{Y}$ *by*

$$Sp^\tau(\mathfrak{Y}) := \{ E \in \mathbb{R} \mid \hat{f}(E) = 0\ \ \forall f \in \mathfrak{I}_{\mathfrak{Y}}^\tau \},$$

where for a singleton $\mathfrak{Y} = \{A\}$ *we write* $Sp^\tau(A)$ *and call it the spectrum of A;*

(iii) *the (total) Arveson spectrum of the W*-dynamical system by*

$$Sp(\tau) := Sp^\tau(\mathfrak{M});$$

(iv) *the spectral subspaces of* $\mathfrak{M}$ *for a* σ*-closed set* $\Sigma \subset \hat{\mathbb{R}}$ *by*

$$\mathfrak{M}^\tau(\Sigma) := \overline{\{ A \in \mathfrak{M} \mid Sp^\tau(A) \subseteq \Sigma \}}^{\sigma},$$

abbreviating henceforth $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ *by* σ*;*

(v) *the set of* τ*-invariant elements*

$$\mathfrak{M}^\tau := \{A \in \mathfrak{M} \mid \tau_t(A) = A, \ \forall t \in \mathbb{R}\}.$$

The ideal property of $\mathfrak{I}_{\mathfrak{Y}}^\tau$ is seen by forming the Fourier transforms. The Arveson spectrum may also be written by the polar set

$$Sp(\tau) = (\ker \overline{\tau})^\circ.$$

We list some properties of the spectrum:

A.2. Lemma

Let be $(\mathfrak{M}, \mathbb{R}, \tau)$ *a W*-dynamical system as above and* $A, B \in \mathfrak{M}$*. Then it holds:*

(i) $Sp^\tau(\tau_t(A)) = Sp^\tau(A) \ \ \forall t \in \mathbb{R}$.

(ii) $Sp^\tau(\alpha A + B) \subseteq Sp^\tau(A) \cup Sp^\tau(B)$.

(iii) $Sp^\tau(\tau_f(A)) \subseteq Sp^\tau(A) \cap supp(\hat{f})$.

(iv) If, for $f, g \in L^1(\mathbb{R})$ such that in a neighbourhood of $Sp^\tau(A)$, $\hat{f} = \hat{g}$, then $\tau_f(A) = \tau_g(A)$.

PROOF: This follows immediately from the definitions.

A.3. Lemma

For a W-dynamical system* $(\mathfrak{M}, \mathbb{R}, \tau)$ *we have the following criteria for the local spectrum of* $A \in \mathfrak{M}$*, following directly from its definition:*

(i) $E \in Sp^\tau(A)$, iff $\hat{f}(E) \neq 0 \Longrightarrow \tau_f(A) \neq 0$, $f \in L^1(\mathbb{R})$.

(ii) *If* B *is a closed subset of* $\hat{\mathbb{R}}$*, then*

$$Sp^\tau(A) \subset B, \text{ iff } supp \hat{f} \cap B = \varnothing \Longrightarrow \tau_f(A) = 0, \ f \in L^1(\mathbb{R}).$$

B. Arveson Spectrum in Hilbert Space

If the locally compact abelian group $\mathbb{R}$ has a strongly continuous, unitary representation

$$U : \mathbb{R} \longrightarrow \mathcal{B}(\mathcal{H})$$

in a Hilbert space $\mathcal{H}$, the Arveson spectral theory can also be applied to this case, dealing quite generally with groups of isometries in a Banach space [12]. In complete analogy to definitions for the dynamical W*-automorphisms we introduce for the unitary dynamics U_f, $Sp(U)$, $Sp^U(\Psi)$ for $\Psi \in \mathcal{H}$, and $\mathcal{H}^U(\Sigma)$ for a closed $\Sigma \subseteq \hat{\mathbb{R}}$.

B.1. Proposition

Let $(\mathfrak{M}, \mathbb{R}, \tau)$ *be a* W^**-dynamical system in a Hilbert space* $\mathcal{H}$ *such that*

$$\tau_t(A) = U_t A U_t^*, \quad \forall t \in \mathbb{R}, \ \forall A \in \mathfrak{M},$$

where $t \longrightarrow U_t$ *is a strongly continuous, unitary representation of* $\mathbb{R}$ *with an invariant vector* $\Omega \neq 0$*. Then there holds:*

(i) *There exists a mapping*

$$P_U : \mathcal{B}(\hat{\mathbb{R}}) \longrightarrow \mathcal{P}(\mathcal{H})$$

from the Borel sets $\mathcal{B}(\hat{\mathbb{R}})$ *of* $\hat{\mathbb{R}}$ *into the set of projections* $\mathcal{P}(\mathcal{H})$ *on* $\mathcal{H}$*, such that*

(a) P_U *is* σ*-additive, with* $P_U(\emptyset) = 0$ *and* $P_U(\hat{\mathbb{R}}) = \mathbb{1}$*;* $P_U(\Sigma_1 \cap \Sigma_2) = P_U(\Sigma_1)P_U(\Sigma_2)$ *for all* $\Sigma_1, \Sigma_2 \in \mathcal{B}(\hat{\mathbb{R}})$.

(b) $\mathcal{H}^U(\Sigma) = P_U(\Sigma)\mathcal{H}$ *for all closed* $\Sigma \subseteq \hat{\mathbb{R}}$.

(c) $U_f = \int_{\mathbb{R}} \hat{f}(E) \, \mathrm{d}\, P_U(E)$ *holds in the strong operator topology for all Fourier transforms* $\hat{f}$ *of bounded measures on* $\mathbb{R}$ *and may be generalized to all bounded measurable functions* $\hat{f}$ *on* $\hat{\mathbb{R}}$*. For the special function* $\hat{f}(E) = \exp(\mathrm{i}Et)$*, which is the Fourier transform of the Dirac measure* $f(t) = \delta(t)$*, we obtain* $U_t = \int_{\mathbb{R}} \exp(\mathrm{i}Et) \, \mathrm{d}\, P_U(E)$.

(d) $E \in Sp(U)$*, iff* $P_U(\Sigma) \neq 0$ *for all closed neighbourhoods* Σ *of* E.

(e) P_U *is the resolution of the identity in Stone's theorem: Defining*

$$K := \int_{\mathbb{R}} E \, \mathrm{d}P_U(E) \quad \text{with} \qquad \mathfrak{D}_K := \{\Psi \in \mathcal{H} \mid \int_{\mathbb{R}} E^2 \langle \Psi | \mathrm{d}\, P_U(E) \Psi \rangle < \infty\}, \tag{B.1}$$

we obtain a selfadjoint operator in the usual way. Then it holds

$$U_f = \hat{f}(K) \tag{B.2}$$

by the definition of a function of K *and . One finds*

$$Sp(U) = \sigma(K), \tag{B.3}$$

where the right hand side denotes the usual spectrum of the selfadjoint operator K.

(ii) $Sp^\tau(A) \supseteq Sp^U(A\Omega)$ *for all* $A \in \mathfrak{M}$.

(iii) $A \in \mathfrak{M}^\tau(\Sigma) \implies A\Omega \in P_U(\Sigma)\mathcal{H}$ for all closed $\Sigma \subseteq \hat{\mathbb{R}}$.

(iv) If Ω is also cyclic for $\mathfrak{M}$, then

$$Sp(\tau) \supseteq Sp(U).$$

(v) *If Ω is cyclic and separating for $\mathfrak{M}$, then also the reverse relations are valid in (ii)–(iv). Especially, it holds*

$$Sp(\tau) = Sp(U).$$

PROOF: For spational limitation we refer to a future publication.

C. The Borchers-Arveson Theorem

C.1. Theorem (Borchers, Arveson)

Let be $(\mathfrak{M}, \mathbb{R}, \tau)$ a W-dynamical system, where $\mathfrak{M}$ is a von Neumann algebra on the Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(i) *There is a strongly continous one-parameter unitary group $t \mapsto U_t \in \mathcal{B}(\mathcal{H})$ with nonnegative spectrum such that*

$$\tau_t(A) = U_t A U_t^*$$

for all $A \in \mathfrak{M},\ t \in \mathbb{R}$.

(ii) *There is a strongly continous one-parameter unitary group $t \mapsto U_t \in \mathfrak{M}$ with nonnegative spectrum such that*

$$\tau_t(A) = U_t A U_t^*$$

for all $A \in \mathfrak{M},\ t \in \mathbb{R}$.

(iii) $\bigcap_{E\in\mathbb{R}} [\mathfrak{M}^\tau([E,\ \infty))\ \mathcal{H}] = 0$, *where [linear subspace of $\mathcal{H}$] denotes the projection onto the closure of the linear subspace.*

Moreover, if (one of) these conditions are satisfied, one may take for the unitary group U the special elements

$$U_t^{BA} = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tE}\ \mathrm{d}P(E),$$

where the spectral measure is obtained as follows: Define

$$Q([E,\ \infty)) := \bigcap_{E'\leq E} [\mathfrak{M}^\tau([E';\ \infty))\ \mathcal{H}]\,,$$

which is a projection in $\mathfrak{M}$, and assume the relation (called 'intersection property' in our text)

$$\lim_{E\to\infty} Q([E,\ \infty)) =: Q_\infty = 0.$$

Define the distinguished partition of unity

$$P((-\infty, E)) := \mathbb{1} - Q([E,\ \infty)), \quad \forall E \in \mathbb{R}.$$

The latter defines the projection valued measure $\mathrm{d}P(\cdot)$ on $\mathbb{R}$, with values in $\mathfrak{M}$, which has been employed in the above spectral representation of U_t^{BA}.

According to the Stone theorem, the generator of the special unitary implementation U_t^{BA} of τ_t has the form

$$H^{BA} = \int_{\mathbb{R}} E \mathrm{d}P(E)$$

and is, of course, affiliated with $\mathfrak{M}$.

Observe, that in the case, where there exists a unitary implementation of the W*-dynamical system with a both-sided unbounded generator, it is not excluded that $Q_\infty = 0$, which in turn then would imply that there exists also a lower bounded implementation. If, however, $Q_\infty > 0$, then all implementations have both-sided unbounded generators, which may or not be, affiliated with $\mathfrak{M}$.

[1] A. L. Fetter and J. D. Walecka, Quantum Theory of Many-particle Systems, McGraw-Hill Book Company, New York 1971.

[2] T. van Duzer and C. W. Turner, Principles of Superconducting Devices and Circuits, Elsevier, New York 1984.

[3] H. Haken, Quantenfeldtheorie des Festkörpers, Teubner, Stuttgart 1973.

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] D. J. Thouless, The Quantum Mechanics of Many Body Systems, Academic Press, New York, London 1961.

[6] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, Tokyo 1975.

[7] G. Rickayzen, Theory of Superconductivity, J. Wiley, New York 1965.
[8] H. Umezawa, Advanced Field Theory, AIP Press, New York 1993.
[9] T. Gerisch and A. Rieckers, Helv. Phys. Acta **70**, 727 (1997).
[10] R. Münzner and A. Rieckers, Green's Functions at Finite and Zero Temperature in Different Ensembles for a Bipolaronic Superconductor, preprint, Tübingen 2004.
[11] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Volume 1, Springer-Verlag, New York 1987.
[12] W. Arveson, J. Funct. Anal. **15**, 217 (1974).
[13] R. Münzner and A. Rieckers, Spectral Properties of Green's Functions for Perturbed Mean Field Models, preprint, Tübingen 2005.
[14] H. J. Borchers, Comm. Math. Phys. **2**, 49 (1966).
[15] L. P. Gor'kov, Soviet Phys.-JETP **9**, 1364 (1959).
[16] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969).
[17] J. von Neumann, Composito Mathematicae **6**, 1 (1938).
[18] M. Takesaki, Theory of Operator Algebras I, Springer, New York 1979.
[19] A. Connes, Ann. Sci. Ecole Norm. Sup. **6**, 133 (1973).
[20] M. Reed and B. Simon, Functional Analysis, Vol. 1, Academic Press, Inc., London 1980.
[21] M. Takesaki, Acta Math. **119**, 273 (1967).
[22] H. Araki, Pac. J. Math. **50**, 309 (1974).
[23] A. Connes, Ann. Inst. Fourier Grenoble **24**, 121 (1974).
[24] U. Haagerup, Math. Scand. **37**, 271 (1974).
[25] S. Doplicher, D. Kastler, and D. W. Robinson, Comm. Math. Phys. **3**, 1 (1966).
[26] G. K. Pedersen, C*-Algebras and their Automorphism Groups, Academic Press, New York 1979.
[27] W. Bös, Inventiones Mathematicae **37**, 241 (1976).
[28] H. Bauer, Wahrscheinlichkeitstheorie, de Gruyter Verlag, Berlin 1991.
[29] H. Bauer, Maß- und Integrationstheorie, de Gruyter Verlag, Berlin 1990.
[30] T. Hida, Brownian Motion, Springer Verlag, New York 1980.